Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems
Spring 2018

Solutions exercise 1

From Jordan and Smith, chapter 1

$\boxed{1}$ Locate the equilibrium points and sketch the phase diagrams in their neighbourhood.

(i) $\ddot{x} - k\dot{x} = 0$ ($k \neq 0$).

Let $\dot{x} = y$, then one obtains the system

$$\dot{x} = y,$$
$$\dot{y} = k\dot{x} = ky.$$

Equilibrium points are given where $\dot{x} = \dot{y} = 0$. In this case, this holds when $y = 0$ and $x \in \mathbb{R}$ is arbitrary.

It follows from the system of equations that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = k.$$

This differential equation has solution $y(x) = kx + C$ for some constant C. We first consider the case $k > 0$. We see that $\dot{y} > 0$ for $y > 0$ and $\dot{y} < 0$ for $y < 0$. The phase paths are oriented away from the x-axis, and thus the equilibrium points are unstable. See Figure 1 for a sketch of the phase diagram. For the case $k < 0$, the phase paths are oriented towards the x-axis and therefore the equilibrium points are stable. Try to sketch the phase diagram for $k > 0$.

(ii) $\ddot{x} - 8x\dot{x} = 0$.

Let $\dot{x} = y$ to obtain $\dot{y} = 8xy$. As in the previous example, the points where $y = 0$ and $x \in \mathbb{R}$ are equilibrium points.

The system has solution $y(x) = 4x^2 + C$ for some constant C. By studying the sign of $\dot{y}$ we see that the equilibrium points are stable for $x \leq 0$ and unstable for $x > 0$. See Figure 2 for a sketch of the phase diagram.

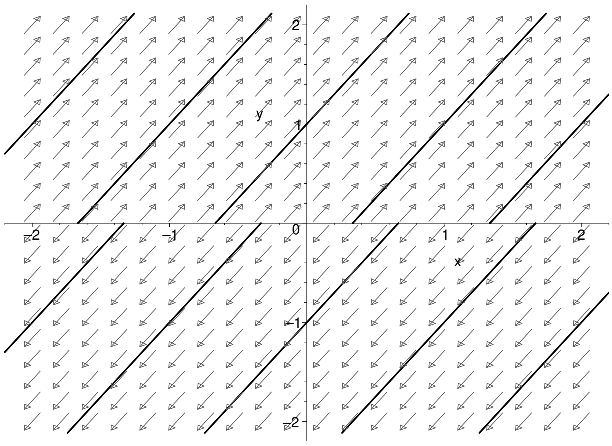

Figure 1: Phase diagram of $\ddot{x} - k\dot{x} = 0$ for $k > 0$

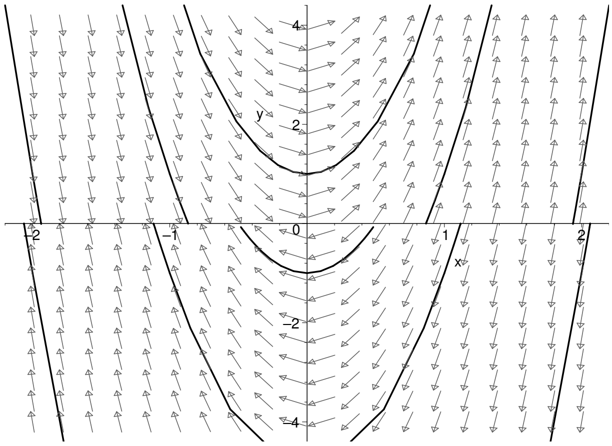

Figure 2: Phase diagram of $\ddot{x} - 8x\dot{x} = 0$

(iii) $\ddot{x} = k$ for $|x| > 1$ and $\ddot{x} = 0$ for $|x| < 1$ $(k \neq 0)$.

Let $\dot{x} = y$ to obtain the system of equations

$$\dot{x} = y,$$
$$\dot{y} = \begin{cases} k & \text{for } |x| > 1 \\ 0 & \text{for } |x| < 1. \end{cases}$$

Note that $\dot{y}$ is zero when $|x| < 1$. When $|x| > 1$, $\dot{y}$ is never zero. Further, $\dot{x} = 0$ when $y = 0$. The equilibrium points are then given by $(\gamma, 0)$ where $\gamma \in (-1, 1)$.

We see that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{y}$, for $y \neq 0$ and $|x| > 1$. This gives $x(y) = \frac{1}{2k}y^2 + C_1$ for some constant C_1. We have that when $y = 0$ and $|x| > 1$, that both x and y are changing and that the corresponding phase paths cross the x-axis. For $|x| < 1$ we get the trivial solution $y(x) = C_2$. See Figure 3 for a sketch of the phase diagram when $k > 0$. Try to analyze the case $k < 0$ by yourself.

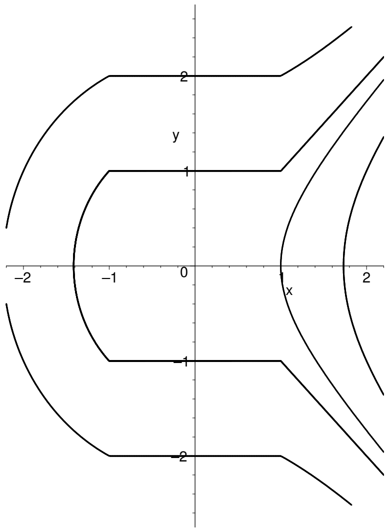

Figure 3: Phase diagram for $\ddot{x} = k$ for $|x| > 1$ and $\ddot{x} = 0$ for $|x| < 1$, where $k > 0$