Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems
Spring 2018

Solutions exercise 4

2.3

(i) Sketch the phase diagram and characterize the equilibrium points of

$$\dot{x} = x - y,$$
$$\dot{y} = x + y - 2xy.$$

Equilibrium points are found by setting $\dot{x} = \dot{y} = 0$, which gives $x = y$ and $x + y - 2xy = 0$. This gives $x = y = 0$, or $x = y = 1$. Hence, $(0,0)$ and $(1,1)$ are equilibrium points.

The matrix for linearization is

$$J = \begin{bmatrix} 1 & -1 \\ 1-2y & 1-2x \end{bmatrix}.$$

Linearization about the origin gives us the equations $\dot{x} = x - y$ and $\dot{y} = x + y$. This system can be solved in the usual way by looking at eigenvectors and eigenvalues. We find $\lambda = 1 \pm i$. Hence, we have an unstable spiral at the origin. The orientation can be found by setting $y = 0$ and $x > 0$. This gives $\dot{y} > 0$, so the spiral has a motion counterclockwise.

Linearization about $(1,1)$ gives $\dot{x} = x - y$ and $\dot{y} = -x - y$. We find the eigenvalues as the roots of $(1-\lambda)(-1-\lambda) - 1 = 0$, so $\lambda = \pm\sqrt{2}$. Two real eigenvalues with opposite signs gives us a saddle point. We find the asymptotes of the family of phase paths by looking at the eigenvectors

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \pm \sqrt{2} \end{bmatrix}.$$

Now, we are able to give a sketch of the phase diagram by giving a sketch of a spiral which spreads out counterclockwise at the origin, and at $(1,1)$ a saddle point. See figure 1 for a sketch of the phase diagram.

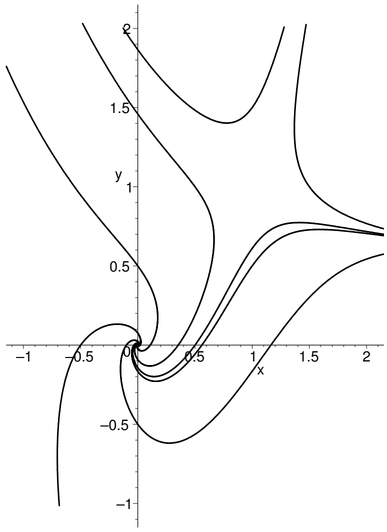

Figure 1: Phase diagram for $\dot{x} = x - y$, $\dot{y} = x + y - 2xy$

(vi) Sketch the phase diagram and characterize the equilibrium points of

$$\begin{aligned}\dot{x} &= -6y + 2xy - 8,\\ \dot{y} &= y^2 - x^2.\end{aligned}$$

At equilibrium we have $\dot{y} = 0$ so that $x = \pm y$. By inserting this into the first equations, we get two possible second order equations for y. Only one of them has real solutions, and the equilibrium points are $(-1, -1)$ and $(4, 4)$.

The matrix for linearization is

$$J = \begin{bmatrix} 2y & -6 + 2x \\ -2x & 2y \end{bmatrix}.$$

We look at the equilibrium point $(-1, -1)$ first. Here,

$$J(-1, -1) = \begin{bmatrix} -2 & -8 \\ 2 & -2 \end{bmatrix}.$$

The eigenvalues of $J(-1, -1)$ are given by $\lambda = -2 \pm 4i$. Since these are complex valued with negative real part, the equilibrium point is a stable spiral.

At the point $(4, 4)$ we find

$$J(4, 4) = \begin{bmatrix} 8 & 2 \\ -8 & 8 \end{bmatrix}.$$

The eigenvalues of $J(4, 4)$ are given by $\lambda = 8 \pm 4i$. Since these are complex valued with positive real part, the equilibrium point is an unstable spiral.

See figure 2 for a sketch of the phase diagram.

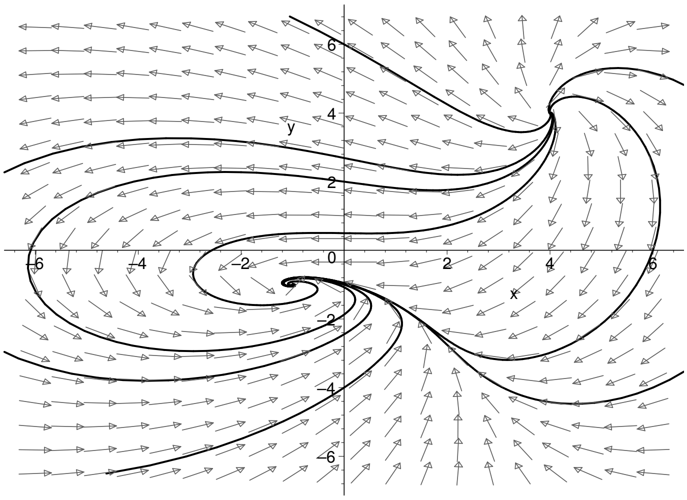

Figure 2: Phase diagram of $\dot{x} = -6y + 2xy - 8$, $\dot{y} = y^2 - x^2$

(ix) Sketch the phase diagram and characterize the equilibrium points of

$$\begin{aligned} \dot{x} &= \sin y, \\ \dot{y} &= -\sin x. \end{aligned} \tag{1}$$

Let $f(x, y) = \sin y$ and $g(x, y) = -\sin x$. Observing that $f_x + g_y = 0$, we deduce that the system (1) is Hamiltonian, and we denote by $H(x, y)$ its Hamiltonian. Recalling that $f = H_y$ and $g = -H_x$. A direct calculation yields $H(x, y) = -(\cos y + \cos x)$. The equilibrium points, which are the solutions to $f(x, y) = 0 = g(x, y)$, are given by $x = m\pi$ and $y = n\pi$ where $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Direct calculations yield

$$q(m\pi, n\pi) = (H_{xx}H_{yy} - H_{xy}^2)(m\pi, n\pi) = \cos(m\pi)\cos(n\pi),$$

and we conclude that

- if both m and n are even, then $q(m\pi, n\pi) > 0$ and the equilibrium points $(m\pi, n\pi)$ are centers;
- if both m and n are odd, then $q(m\pi, n\pi) > 0$ and the equilibrium points $(m\pi, n\pi)$ are centers;
- otherwise $q(m\pi, n\pi) < 0$ and the equilibrium points $(m\pi, n\pi)$ are saddles.

Since the phase diagram is 2π periodic both in the x and the y-direction, it suffices to first draw the phase diagram for $[-\pi, \pi] \times [-\pi, \pi]$ before covering the plane with copies. See figure 3 for a sketch of the phase diagram.

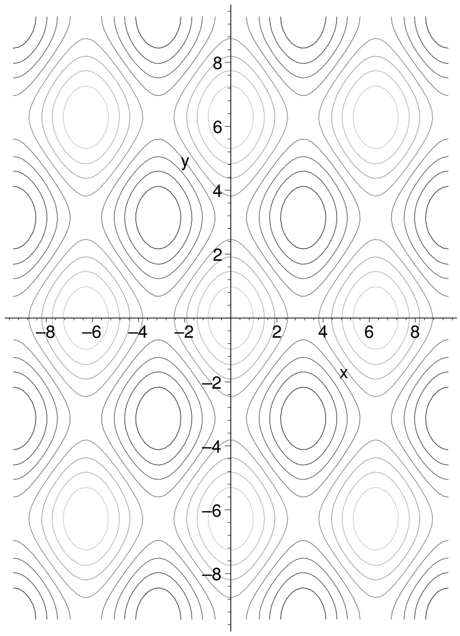

Figure 3: Phase diagram of $\dot{x} = \sin y$, $\dot{y} = -\sin x$

$\boxed{\text{Ex 2013.2}}$ Given H, one can find a dynamical system with H as its Hamiltonian as follows:

$$f(x,y) := \begin{cases} \dot{x} & = H_y = -\cos x \sin y \\ \dot{y} & = -H_x = \sin x \cos y \end{cases}.$$

To show that $(0,0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$ are equilibrium points, we simply plug those values into f:

$$f(0,0) = (-\cos 0 \sin 0, \sin 0 \cos 0) = (0,0),$$

and

$$f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \left(-\cos\frac{\pi}{2}\sin\frac{\pi}{2}, \sin\frac{\pi}{2}\cos\frac{\pi}{2}\right) = (0,0),$$

so both points are equilibrium points.

To classify these, we use the second derivative test.

$$D^2H = \begin{bmatrix} -\cos x \cos y & \sin x \sin y \\ \sin x \sin y & -\cos x \cos y \end{bmatrix}$$

Evaluating at each of the points gives

$$D^2H(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

with determinant $\det D^2H(0,0) = 1$, so $(0,0)$ is a center; and

$$D^2H\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with determinant $\det D^2H\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -1$, so $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a saddle.