Norwegian University of Science and Technology Department of Mathematical Sciences TMA4165 Differential equations and dynamical systems Spring 2018

Solutions exercise 5

2.3

(ii) Sketch the phase diagram and characterize the equilibrium points of

$$\dot{x} = ye^y,$$
  
$$\dot{y} = 1 - x^2$$

Equilibrium points are found by setting  $\dot{x} = 0$  og  $\dot{y} = 0$ , which gives  $x = \pm 1$  and y = 0. Hence  $(\pm 1, 0)$  are equilibrium points.

We linearize the system about the points  $(\pm 1, 0)$ .

Near (1,0) we have  $\dot{x} = y$  and  $\dot{y} = -2(x-1)$ . Hence

$$\frac{dy}{dx} = -2\frac{x-1}{y}.$$

Solving the differential equation gives an ellipse around the point (1,0)

$$2(x-1)^2 + y^2 = C,$$

and hence, a centre. Note that a centre for the linearized system is not necessarily a centre in the original system. In the original system, we might have a spiral (stable or unstable).

The system is Hamiltonian (since  $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0$  where  $\dot{x} = X(x, y)$  and  $\dot{y} = Y(x, y)$ ) which can be used to decide whether (1, 0) is a centre or not. If the Hamiltonian function has a local maximum or minimum at the point (1, 0), the equilibrium point (1, 0) is a centre, also in original system.

We have  $\dot{x} = \frac{\partial H}{\partial y}$  and  $\dot{y} = -\frac{\partial H}{\partial x}$ , so that

$$\frac{\partial H}{\partial x} = x^2 - 1,$$
$$\frac{\partial H}{\partial y} = ye^y.$$

If we integrate the last equation with respect to y we get

$$H(x, y) = e^{y}(y - 1) + f(x)$$

which, inserted into the equation  $\frac{\partial H}{\partial x} = x^2 - 1$  gives

$$\frac{df}{dx} = x^2 - 1$$

By integrating this equation with respect to x we get  $H(x, y) = e^{y}(y-1) + \frac{1}{3}x^{3} - x$ . Then we can use the second derivative test to prove that H(x, y) has a local maximum or minimum at (1, 0). This shows that (1, 0) is a centre in the original system. The direction of the paths are clockwise, since  $\dot{y} < 0$  for y = 0 and x > 1.

Linearization about (-1, 0) gives  $\dot{x} = y$  and  $\dot{y} = 2(x+1)$ . Hence

$$\frac{dy}{dx} = 2\frac{x+1}{y}$$

The paths are hyperbolas centered at (-1,0). We find the asymptotes of the hyperbolas,  $y = \pm \sqrt{2}(x+1)$ . The direction along the asymptotes are found by studying the sign of  $\dot{y}$  in each quadrant. We may also use the direction of the paths about (1,0) to sketch the direction along the asymptotes. See figure 1 for a sketch of the phase diagram.



Figure 1: Phase diagram of  $\dot{x} = ye^y$ ,  $\dot{y} = 1 - x^2$ 

(iii) Sketch the phase diagram and characterize the equilibrium points of

$$\dot{x} = 1 - xy,$$
  
$$\dot{y} = (x - 1)y$$

We find the equilibrium points of the system by setting  $\dot{x} = \dot{y} = 0$ , which gives x = y = 1 as the equilibrium point.

We linearize the system about the point (1, 1) which gives us the linearized system

$$\dot{x} = -x - y$$
$$\dot{y} = x.$$

The eigenvalues of the matrix of the system are given as solutions to  $(-1-\lambda)(-\lambda) + 1 = \lambda^2 + \lambda + 1 = 0$ . Hence

$$\lambda = \frac{-1 \pm \sqrt{3}i}{2}.$$

These are complex valued eigenvalues with negative real part. This is a stable spiral locally around (x, y) = (1, 1). The direction is counterclockwise since  $\dot{y} > 0$  for x > 0 and y = 0. Se figure 2 for a sketch of the phase diagram.



Figure 2: Phase diagram of  $\dot{x} = 1 - xy$ ,  $\dot{y} = (x - 1)y$ 

2.9 Find the equations for the phase paths for the general epidemic described by the system

$$\begin{split} \dot{x} &= -\beta xy, \\ \dot{y} &= \beta xy - \gamma y, \\ \dot{z} &= \gamma y. \end{split}$$

Sketch the phase diagram in the x, y plane. Confirm that the number of infectives reaches its maximum when  $x = \frac{\gamma}{\beta}$ .

We find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\beta xy - \gamma y}{-\beta xy} = -1 + \frac{\gamma}{\beta} \frac{1}{x},$$

for x > 0. Integration then gives

$$y(x) = -x + \frac{\gamma}{\beta} \ln x + C.$$

We find the maximum of the number of infectives by setting

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1 + \frac{\gamma}{\beta}\frac{1}{x} = 0,$$

hence

$$x = \frac{\gamma}{\beta}.$$

To verify that this is infact a maximum, we see that the sign of the double derivative of y is negative. The solution for z is given by  $z = -x_0 - y_0 + z_0 - x - y$ .

See figure 3 for a sketch of the phase diagram.



Figure 3: Phase diagram of  $\dot{x} = -\beta xy$ ,  $\dot{y} = \beta xy - \gamma y$ . Here,  $\beta = 2$  and  $\gamma = 10$ . The total population is set to 10 individuals.

Exam 2011, 5 (a)

We are given the system

with

$$\mathbf{f}(x,y) = \begin{bmatrix} 2y(x^4 - 2x^2 + 2) \\ 4(x - x^3)(y^2 + 1) \end{bmatrix}.$$

 $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ 

The divergence of f is

$$\operatorname{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 2y(4x^3 - 4x) + 4(x - x^3)2y = 0,$$

so the system is Hamiltonian. The Hamilton function satisfies

$$\frac{\partial H}{\partial y} = f_1$$
 and  $-\frac{\partial H}{\partial x} = f_2.$ 

Hence,

$$H(x,y) = \int_0^y f_1(x,s)s + C(x) = y^2(x^4 - 2x^2 + 2) + C(x),$$

and

$$f_2 = -\frac{\partial H}{\partial x} = -y^2(4x^3 - 4x) - C'(x),$$

so that

$$C'(x) = -4(x - x^3) \implies C(x) = -2x + x^4 + K$$

By choosing K = 2 we get

$$H(x,y) = (y^{2} + 1)(x^{4} - 2x^{2} + 2).$$

(b)

There are three equilibrium points: (0,0), (1,0) og (-1,0). Since the equilibrium points for f are critical points for H, we can classify them by using the second derivative test for H.

$$D^{2}H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} (12x^{2} - 4)(y^{2} + 1) & 8y(x^{3} - x) \\ 8y(x^{3} - x) & 2(x^{4} - 2x^{2} + 2) \end{bmatrix}.$$

For the point (0,0) we have

$$D^2H(0,0) = \begin{bmatrix} -4 & 0\\ 0 & 4 \end{bmatrix},$$

 $\det D^2 H(0,0) = (-4) \cdot 4 < 0$ , so (0,0) is a saddle point for H and for the dynamical system. For the points  $(\pm 1,0)$  we find

$$D^2 H(\pm 1,0) = \begin{bmatrix} 8 & 0\\ 0 & 2 \end{bmatrix},$$

 $\det D^2 H(\pm 1,0) = 8 \cdot 2 > 0$ , so  $(\pm 1,0)$  are minimium points for H and they are centres for the dynamical system.

Exam 1999, 4 A dynamical system in polar coordinates is given by

$$\dot{\theta} = 1, \quad \dot{r} = \begin{cases} r^2 \sin(\frac{1}{r}) & \text{for } r > 0\\ 0 & \text{for } r = 0. \end{cases}$$
 (1)

Determine if the origin is a stable, asymptotically stable or unstable equilibrium point. Sketch the phase diagram nearby the origin.

For r > 0,  $\dot{r} = 0$  when  $r = \frac{1}{n\pi}$  which gives periodic paths. Look at the interval

$$n\pi < \frac{1}{r} < (n+1)\pi.$$

Here,  $\dot{r} < 0$  if n is odd and  $\dot{r} > 0$  if n is even. Hence, the phase diagram will consist of several limit cycles around the origin, which are closer to each other when r decreases towards zero. Every other cycle is an unstable limit cycle.

After a small perturbation away from the origin, the system will stay in one of the stable limit cycle. The origin is thus a stable equilibrium state, but not asymptotically stable. See figure 4 for a sketch of the phase diagram.



Figure 4: Phase diagram of  $\dot{\theta} = 1$ ,  $\dot{r} = r^2 \sin\left(\frac{1}{r}\right)$  for r > 0,  $\dot{r} = 0$  for r = 0