Norwegian University of Science and Technology
Department of Mathematical Sciences

# TMA4165 Differential equations and dynamical systems Spring 2018

**Solutions exercise 6**

$\boxed{8.11}$ The system

$$\dot{x} = -t^2 x$$
$$\dot{y} = -ty$$

has solution

$$x(t) = x_0 e^{-\frac{t^3}{3}}, \quad y(t) = y_0 e^{-\frac{t^2}{2}}.$$

We show first that the zero solution to the system is asymptotically stable. It is Liapunov stable for all $t \geq 0$, since

$$||\mathbf{x}(t) - \mathbf{0}||^2 = x_0^2 e^{-\frac{2t^3}{3}} + y_0^2 e^{-t^2} \leq x_0^2 + y_0^2 = ||\mathbf{x}(0) - \mathbf{0}||^2.$$

The zero solution is also asymptotically stable,

$$||\mathbf{x}(t) - \mathbf{0}|| = \sqrt{x_0^2 e^{-\frac{2t^3}{3}} + y_0^2 e^{-t^2}} \to 0$$

as $t \to \infty$. By theorem 8.1, all solutions of the system are asymptotically stable.

$\boxed{8.14}$ To find a fundamental matrix for the system

$$\dot{x} = y$$
$$\dot{y} = -x - 2y$$

we write it in the form $\dot{\mathbf{x}} = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

The eigenvalues $A$ are given by the solutions of

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0.$$

We have a double root, $\lambda = -1$. There is only one linearly independent eigenvector:

$$(A - \lambda I)\mathbf{v} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{v} = 0,$$

namely $\mathbf{v} = [1, -1]^T$. The corresponding solution is $[e^{-t}, -e^{-t}]$. To find a second linearly independent solution, write

$$\mathbf{x} = \mathbf{v}te^{-t} + \mathbf{w}e^{-t}$$

so that

$$\dot{\mathbf{x}} - A\mathbf{x} = -[(A+I)\mathbf{w} - \mathbf{v}I]e^{-t} - (A+I)\mathbf{v}te^{-t} = 0.$$

Both terms must be zero in the above equation. The second term is zero by the choice of $\mathbf{v}$. We choose $\mathbf{w}$ so that

$$(A+I)\mathbf{w} = \mathbf{v}$$

resulting in, for example $\mathbf{w} = [0, 1]^T$. Hence, we can write

$$\mathbf{x}(t) = \begin{bmatrix} te^{-t} \\ (1-t)e^{-t} \end{bmatrix}.$$

A fundamental matrix for the system is given by

$$\Phi(t) = \begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{bmatrix}.$$

To find the fundamental matrix $\Psi$ satisfying $\Psi(0) = I$ we calculate

$$\begin{aligned} \Psi(t) =& \Phi(t)\Phi^{-1}(0) = \begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ =& \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}. \end{aligned}$$

**Exam 2016.2** See "previous exams".

**A1**

a)

Suppose

$$\begin{aligned} c_1\cos(t) - c_2\sin(t) &= 0 \\ c_1\sin(t) + c_2\cos(t) &= 0. \end{aligned}$$

For all $t \geq 0$. Then, in particular for $t = 0$ we find $c_1 = c_2 = 0$ so the two vectors are linearly independent.

b) We note that $2\mathbf{x_1} = \mathbf{x_2}$ so the vectors are linearly dependent.

c) If

$$
\begin{aligned}
c_1 e^t + c_2 t e^t &= 0 \\
2c_1 e^t + (2t+1) c_2 e^t &= 0
\end{aligned}
$$

for all $t \geq 0$. In particular, for $t = 0$ the first equation gives $c_1 = 0$. Inserted into the second equation gives $c_2 = 0$ so the vectors are linearly independent.

d) Differentiating $\mathbf{x}_1$ and $\mathbf{x}_2$ one obtains

$$
A = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}.
$$

A close look at $\mathbf{x}_1$ and $\mathbf{x}_2$ reveals that $A$ should have as eigenvalues $\lambda_1 = \lambda_2 = 1$. Moreover, the polynomial $2t+1$ in the second component of $\mathbf{x}_2$ indicates that one can only find one linearly independent eigenvector. This can be justified by computing both the eigenvalues and eigenvectors of $A$.

e) Assume

$$
\begin{aligned}
c_1 e^{2t} + c_2 t e^t &= 0 \\
2c_1 e^{2t} + (2t+1) c_2 e^t &= 0.
\end{aligned}
$$

The first equation gives $c_1 = -c_2 t e^{-t}$. Inserted into the second equation gives

$$
-2c_2 t e^t + (2t+1) c_2 e^t = c_2 e^t = 0,
$$

which implies $c_2 = 0$ and then also $c_1 = 0$. Hence, the vectors are linearly independent.

f) Let

$$
c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = \mathbf{0}.
$$

This holds for any $t \in \mathbb{R}$ if and only if $c_1 = c_2 = 0$. Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.