Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2018

Solutions exercise 7

1992,2

a) The system can be written
$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{h}(\mathbf{x}) \tag{1}$$
where
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -5 \end{bmatrix},$$
$$\mathbf{h}(\mathbf{x}) = \begin{bmatrix} -xy^2 - x^3 \\ 3x^2y - 2yz^2 - y^3 \\ y^2z - z^3 \end{bmatrix}.$$

The zero solution of $\dot{\mathbf{x}} = A\mathbf{x}$ are asymptotically stable since all the eigenvalues of A are negative. Since $\mathbf{h}(\mathbf{x}) = O(|\mathbf{x}|^2)$, the zero solution of the system (1) is an asymptotically stable equilibrium point.

b) Choosing a Liapunov function $V\big(x(t), y(t), z(t)\big)$ as
$$V\big(x(t), y(t), z(t)\big) = 3x^2(t) + y^2(t) + 2z^2(t),$$
one calculates that
$$\frac{\mathrm{d}V}{\mathrm{d}t} = -6x^2 - 6x^4 - 14y^2 - 2y^4 - 20z^2 - 4z^4.$$
It follows that
$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -2V,$$
which gives rise to
$$V\big(x(t), y(t), z(t)\big) \leq V\big(x(0), y(0), z(0)\big)\exp(-2t).$$
This means that each solution tends (exponentially) to the origin as $t \to +\infty$.

1993,1 Given the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

Find, with the help of the matrix exponential, the solution which satisfies $x(0) = y(0) = z(0) = 1$.

The solution of equation (2) is given by

$$\mathbf{x} = e^{At}\mathbf{x}_0$$

where $\mathbf{x}_0$ is the initial condition. Here,

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = 2 \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{B} = 2I + B.$$

Note that

$$B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{og} \quad B^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so that $B^m = 0$ for $m \geq 3$. This means that B is *nilpotent*. Using this,

$$\begin{aligned} \mathbf{x} &= e^{At}\mathbf{x}_0 = e^{(2I+B)t}\mathbf{x}_0 = e^{2It}e^{Bt}\mathbf{x}_0 \\ &= e^{2t}I\left(I + Bt + \frac{B^2t^2}{2}\right)\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e^{2t}\begin{bmatrix} 1 + t + \frac{t^2}{2} \\ 1 + t \\ 1 \end{bmatrix}. \end{aligned}$$

1995,5

a) We use polar coordinates and find

$$\begin{aligned} \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} = 3r, \\ \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} = 2. \end{aligned}$$

When we integrate this system we get

$$r(\theta) = Ce^{\frac{3}{2}\theta}$$

where C is a constant. If we start at $(x_0, 0)$, we see that $C = x_0$ and after one round, we have

$$r(2\pi) = x_0e^{3\pi}.$$

b) From the analysis in a) we see that the equilibrium point is an unstable spiral. We can also calculate the eigenvalues $\lambda = 3 \pm 2i$ to confirm this. For the second system, we have

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{G}(\mathbf{x})$$

where A is the same as for the system in a) and $\mathbf{G}(\mathbf{x}) = O(|\mathbf{x}|^2)$. Hence, the second system also has an unstable spiral at $(0, 0)$.

2013.5 See "previous exams".