Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2018

Solutions exercise 10

$\boxed{3.7}$ A limit cycle encloses N nodes, F spirals, C centres, and S saddle points. Show that

$$N + F + C - S = 1.$$

Let e be the limit cycle in question. Then $I_e = 1$. By theorem 3.2 in [JS] we have

$$1 = I_e = \sum_{i=1}^{n} I_i = N + F + C - S$$

since the index for a saddle point is -1 and for nodes, spirals and centres we have $I = 1$.

$\boxed{3.12}$ We are to find the index at infinity for the system

$$\dot{x} = x - y$$
$$\dot{y} = x - y^2.$$

Here, there are two equilibrium points, namely $(0,0)$ and $(1,1)$. The matrix of linearization is given by

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -2y \end{bmatrix}$$

at the point (x, y).

The eigenvalues of A at the origin can be found by solving the system $-\lambda(1-\lambda)+1 = 0$ which gives $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. This is an unstable spiral also in the original system and so the index at the origin is $I(0,0) = 1$.

The eigenvalues of A at $(1,1)$ can be found by solving $(1-\lambda)(-2-\lambda)+1 = 0$. This gives $\lambda = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. This is a saddle point, so $I(1,1) = -1$.

The index at infinity is now given by J.S. theorem 3.4,

$$I_\infty = 2 - (1 - 1) = 2.$$

$\boxed{3.25}$ Show that the second order system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ and $\operatorname{curl}(\mathbf{X}) = 0$ in a simply connected region D cannot have closed paths.

First note that

$$\operatorname{curl}(\mathbf{X}) = 0 \quad \Leftrightarrow \quad \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}.$$

We solve the problem using Green's theorem. Suppose we have a closed curve Γ and call the interior region R. Then we have

$$0 = \iint_R \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \mathrm{d}A = \oint_\Gamma \mathbf{X} \cdot \mathbf{T} \, \mathrm{d}s$$

where $\mathbf{T}$ is the unit tangent vector for the curve Γ. However, since Γ is a phase path, $\mathbf{T}$ will always be parallel to the vector $\mathbf{X}$. Hence, the right hand side in the previous equation cannot be zero and we have a contradiction.

The system

$$\begin{aligned} \dot{x} &= y + 2xy, \\ \dot{y} &= x + x^2 - y^2 \end{aligned}$$

satisfy the conditions, so it follows that the system does not have any closed paths.

$\boxed{8.3}$ Find the limit cycles of the system

$$\begin{aligned} \dot{x} &= -y + x \sin \sqrt{x^2 + y^2}, \\ \dot{y} &= x + y \sin \sqrt{x^2 + y^2}. \end{aligned}$$

Determine which of them are Poincaré stable.

Introduce polar coordinates to find

$$\begin{aligned} \dot{r} &= r \sin r, \\ \dot{\theta} &= 1. \end{aligned}$$

Since $\dot{r}$ is only a function of r, we have limit cycles where $\dot{r} = 0$. This gives us limit cycles when $r = n\pi$ where $n \in \mathbf{N}$. For limit cycles of the form $r = (2n+1)\pi$, $\dot{r}$ is positive just inside the limit cycle and negative just outside the limit cycle. This is a stable limit cycle. The other limit cycles are unstable. See figure 1 for a sketch of the phase diagram.

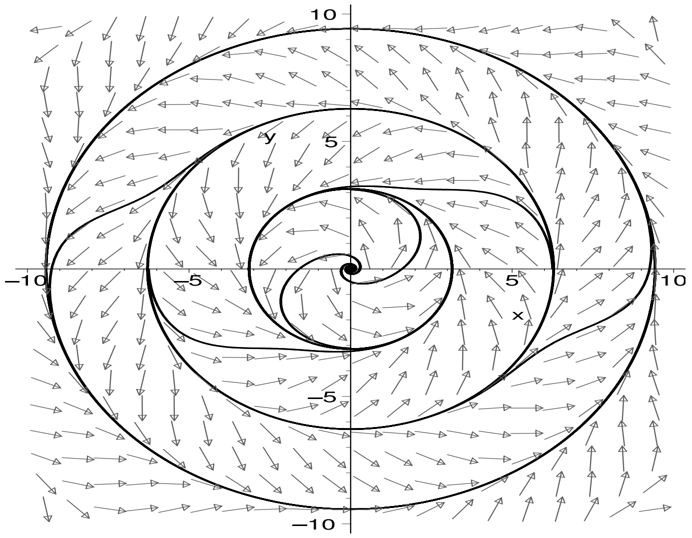

Figure 1: Phase diagram of $\dot{x} = -y + x\sin\sqrt{x^2+y^2}$, $\dot{y} = x + y\sin\sqrt{x^2+y^2}$