Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2018

Solutions exercise 11

11.5 Show that the origin is a centre for the equations

$$\ddot{x} - x\dot{x} + x = 0,$$
$$\ddot{x} + x\dot{x} + \sin x = 0.$$

The first equation may be written

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x) = -x$ and $g(x) = x$. Both f and g are odd functions, $f(x) < 0$ for $x > 0$, $g(x) > 0$ for $x > 0$, and

$$g(x) = x > \alpha f(x) \int_0^x f(u)\mathrm{d}u = \alpha \frac{x^3}{2}$$

for a fixed $\alpha > 1$ if we are close enough to $x = 0$. For example, if we choose $\alpha = 4$ the equation holds for $x < \frac{1}{2}$. By theorem 11.3, the origin is a centre.

Similarly, we can write the second equation with $f(x) = x$ and $g(x) = \sin(x)$. Both f and g are odd functions, f does not change sign for positive x, and

$$g(x) = \sin(x) > \alpha f(x) \int_0^x f(u)\mathrm{d}u = \alpha \frac{x^3}{2}$$

for $\alpha > 1$ and $0 < x < \epsilon$ if we choose ϵ small enough. In this domain, we also have $g(x) > 0$ for $x > 0$. By theorem 11.3, the origin is a centre.

1996,1 Given the system

$$\dot{x} = x - y$$
$$\dot{y} = x^2 - 1.$$

a) Find and classify all equilibrium points of the system. Sketch the phase diagram.

b) Does there exist a closed phase path surrounding all equilibrium points?

a) The equilibrium points are given when $x = y$ and $x^2 - 1 = 0$. Hence, the equilibrium points are $(-1,-1)$ and $(1,1)$. The matrix of linearization is given by

$$J = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}.$$

At the point $(1,1)$ we find $\lambda = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$. Hence, the point $(1,1)$ is an unstable spiral. At the point $(-1,-1)$ we find $\lambda_\pm = \frac{1}{2} \pm \frac{3}{2}$, so $(-1,-1)$ is a saddle point. The corresponding eigenvectors are

$$v_+ = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_- = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

See figure 1 for a sketch of the phase diagram.

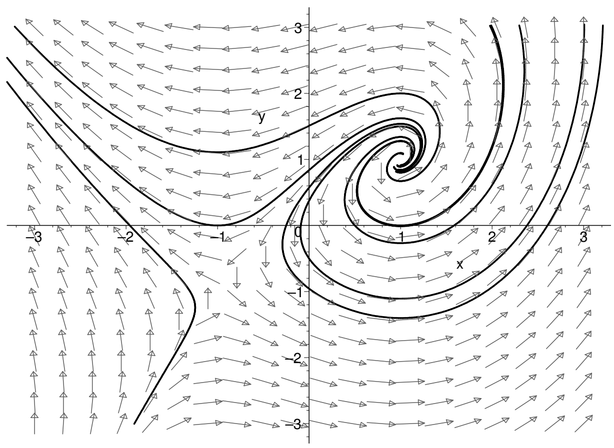

Figure 1: Phase diagram of $\dot{x} = x - y$, $\dot{y} = x^2 - 1$

b) The index of a curve surrounding both the equilibrium points found in **a)** is 0. Closed paths have index $I = 1$, whence there are no closed path surrounding all equilibrium points. Alternatively, by Bendixson's negative criterion:

$$\frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}\left(x^2 - 1\right) = 1.$$

This does not change sign in $\mathbf{R}^2$ so the there are no closed paths.

$\boxed{\text{Exam 1996, 6}}$ Compute the index of the origin for the following systems

a)

$$\dot{x} = x$$
$$\dot{y} = -y.$$

b)

$$\dot{x} = x + x^4 + y^5$$
$$\dot{y} = -y + xy^3.$$

a) Written out in matrix form, the system is

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x$$

The matrix has eigenvalues 1 and -1, so the origin is a saddle point. The index at the origin is $I = -1$.
b) The matrix found in **a)** is the linearization of this system since $x^4 + y^5 = O(|x|^4)$ and $xy^3 = O(|x|^4)$. Hence, $(0,0)$ is a saddle point and $I = -1$.

2002,3

a) State Bendixson's negative criterion.

b) Determine whether or not the following system has non-constant periodic solutions.

$$\dot{x} = y$$
$$\dot{y} = -x - y(1 + x^2 + x^4).$$

c) Given the population model

$$\dot{x} = xF(x,y)$$
$$\dot{y} = yG(x,y),$$

where F and G are C^1 functions. Assume that $\frac{\partial F}{\partial x} < 0$ and $\frac{\partial G}{\partial y} < 0$. Show that there are no closed phase paths in the first quadrant.

a)
Bendixsons negative criterion says that given $\dot{x} = X(x,y)$ and $\dot{y} = Y(x,y)$, if

$$\nabla(X,Y) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

is of one sign in a simply connected domain, there are no periodic paths.

b) We calculate

$$\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}\left(-x - y(1 + x^2 + x^4)\right) = 1 + x^2 + x^4 > 0.$$

By Bendixson's negative criterion, there cannot exists a closed path.
c) We use exercise 3.23 with $\rho(x,y) = \frac{1}{xy}$. Then

$$\frac{\partial}{\partial x}(\rho x F) + \frac{\partial}{\partial y}(\rho y G) = \frac{1}{y}F_x + \frac{1}{x}G_y < 0$$

in the first quadrant, which shows that there are no closed paths.