Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4165 Differential equations and dynamical systems Spring 2018

Solutions exercise 12

Exam 1996, 6 Compute the index of the origin for the following systems

a)

$$\dot{x} = x$$
$$\dot{y} = -y.$$

b)

$$\dot{x} = x + x^4 + y^5$$
$$\dot{y} = -y + xy^3.$$

a) Written out in matrix form, the system is

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x$$

The matrix has eigenvalues 1 and -1, so the origin is a saddle point. The index at the origin is $I = -1$.
b) The matrix found in **a)** is the linearization of this system since $x^4 + y^5 = O(|x|^4)$ and $xy^3 = O(|x|^4)$. Hence, $(0,0)$ is a saddle point and $I = -1$.

Exam 1999, 5 Given the system

$$\dot{x} = x + y - x\sqrt{x^2 + y^2}$$
$$\dot{y} = -x + y - y\sqrt{x^2 + y^2}. \tag{1}$$

a) Classify the equilibrium point $(0,0)$ for both (1) and its linearisation.

b) Show that the system has exactly one closed phase path.

c) Define what it means to be a Poincaré map with Poincaré section Σ.

d) Determine the Poincaré map with Poincaré section $\Sigma = \{(x,0) \mid x > 0\}$.

a) Since $x\sqrt{x^2+y^2} = O(|x|^2), y\sqrt{x^2+y^2} = O(|x|^2)$ we have the linearization

$$\dot{z} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} z$$

at the origin. The matrix has eigenvalues $\lambda = 1 \pm i$. This is an unstable spiral in both the original and the linear system.

b) We use polar coordinates, $r = x^2 + y^2$, $\tan\theta = \frac{y}{x}$ to solve the problem. We find

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y} = 2(x^2+y^2) - 2(x^2+y^2)\sqrt{x^2+y^2} = 2r^2(1-r)$$

and

$$\dot{\theta} = \frac{\dot{y}x - \dot{x}y}{r^2} = -1.$$

We see that we have a periodic solution when $r = 1$, the unit circle. Further, this is the only periodic solutions, since for $r < 1$ we have $\dot{r} > 0$ and for $r > 1$, $\dot{r} < 0$. See figure for a sketch of the phase diagram.

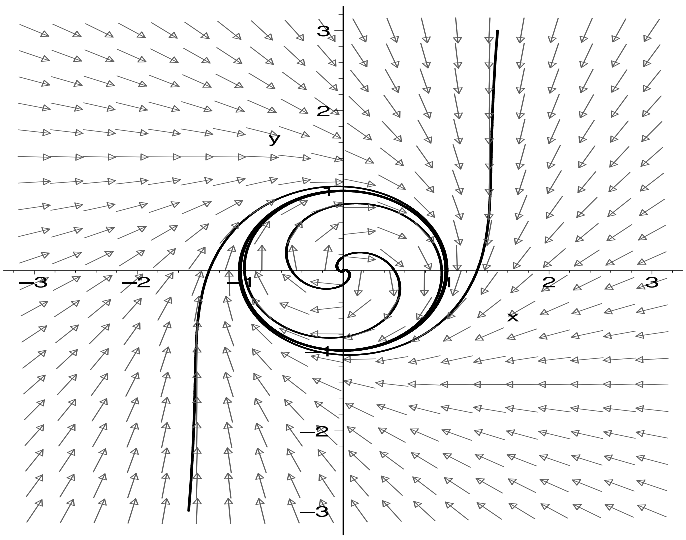

Figure 1: Phase daigram of $\dot{x} = x + y - x\sqrt{x^2+y^2}$, $\dot{y} = -x + y - y\sqrt{x^2+y^2}$

c) Let Σ be a curve or cross section of the $(x, y)-$ plane with the property that it cuts each phase path transversly in some region of the phase diagram, so that it is nowhere tangential to a phase path. Then Σ is called the Poincaré section of the phase diagram. If A_0 is a point on Σ, we follow the phase path through A_0 in the direction of flow until it again hits Σ in a point A_1. This point is called the Poincaré-map of A_0.

d) We solve the differential equations

$$\dot{r} = r(1-r),$$
$$\dot{\theta} = -1.$$

We have

$$\int_{r_0}^{r} \frac{\mathrm{d}\hat{r}}{\hat{r}(1-\hat{r})} = \int_0^t \mathrm{d}\hat{t} \quad \text{which gives} \quad r(t) = \frac{r_0}{r_0 + (1-r_0)e^{-t}}$$

and

$$\int_{\theta_0}^{\theta} \mathrm{d}\hat{\theta} = \int_0^t \mathrm{d}\hat{t} \quad \text{which gives} \quad \theta(t) = -t + \theta_0.$$

With $\theta_0 = 0$ we get $\theta(t) = -t$ so that

$$r(\theta) = \frac{r_0}{r_0 + (1-r_0)e^{\theta}}.$$

For the given Poincaré section we have $r_0 = x > 0$. The Poincaré-map is then

$$P(x) = r(-2\pi) = \frac{x}{x + (1-x)e^{-2\pi}}.$$

Exam 1992, 3 Give an example of an n-dimensional, dynamical system (n given and $n \geq 2$)

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

such that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, $\lim_{t\to\infty} x(t) = 0$ for all solutions, and not all eigenvalues of its linearisation at 0 have strictly negative real part.

The system $\dot{x} = f(x)$ must have a linearization such that some eigenvalues has real-part zero, while the remaining eigenvalues are less than zero. For the linearization, put

$$\dot{x} = Ax$$

where $A = a_{ij}$ is given by

$$\begin{aligned} a_{ii} &= \lambda_i \quad \text{for } i \geq 3 \\ a_{12} &= 1 \\ a_{21} &= -1 \\ a_{ij} &= 0 \quad \text{otherwise.} \end{aligned}$$

That is, we solve the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \\ \dot{x}_i &= \lambda_i x_i. \end{aligned}$$

for $i \geq 3$. We manipulate the first two equations to get a nonlinear system. Put

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 f(r) \\ \dot{x}_2 &= -x_1 + x_2 f(r). \end{aligned}$$

We have

$$\left(\dot{r}^2\right) = 2r^2 f(r).$$

We now choose $f(r)$ so that $\dot{r} < 0$, that is $f(r) < 0$ for all r. This is needed to ensure $\lim_{t\to\infty} x(t) = 0$. A suitable choice is $f(r) = -r^2$, and we get the system

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2)\\ \dot{x}_i &= \lambda_i x_i\end{aligned}$$

for $i \geq 3$.

11.5 Show that the origin is a centre for the equations

$$\begin{aligned}\ddot{x} - x\dot{x} + x &= 0,\\ \ddot{x} + x\dot{x} + \sin x &= 0.\end{aligned}$$

The first equation may be written

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x) = -x$ and $g(x) = x$. Both f and g are odd functions, $f(x) < 0$ for $x > 0$, $g(x) > 0$ for $x > 0$, and

$$g(x) = x > \alpha f(x) \int_0^x f(u)\mathrm{d}u = \alpha \frac{x^3}{2}$$

for a fixed $\alpha > 1$ if we are close enough to $x = 0$. For example, if we choose $\alpha = 4$ the equation holds for $x < \frac{1}{2}$. By theorem 11.3, the origin is a centre.

Similarly, we can write the other equation with $f(x) = x$ and $g(x) = \sin(x)$. Both f and g are odd functions, f does not change sign for positive x, and

$$g(x) = \sin(x) > \alpha f(x) \int_0^x f(u)\mathrm{d}u = \alpha \frac{x^3}{2}$$

for $\alpha > 1$ and $0 < x < \epsilon$ if we choose ϵ small enough. In this domain, we also have $g(x) > 0$ for $x > 0$. By theorem 11.3, the origin is a centre.

11.8 Show that $\ddot{x} + \beta(x^2 - 1)\dot{x} + x^3 = 0$ has one and only one periodic solution. (We have to assume $\beta > 0$ even though this was not mentioned in the exercise. For $\beta < 0$ the system will have an unstable spiral at the origin.)

We can write the equation as

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x) = \beta(x^2 - 1)$ and $g(x) = x^3$. If we define

$$F(x) = \int_0^x f(u)\mathrm{d}u = \beta x\left(\frac{x^2}{3} - 1\right),$$

we see that F is odd, $F(x) = 0$ if and only if $x = 0$ and $x = \pm\sqrt{3}$, F tends to infinity when x tends to ∞, and g is an odd function satisfying $g(x) > 0$ for $x > 0$. The conditions in theorem 11.4 are satisfied, so the equation has a unique periodic solution.

11.9 Show that $\ddot{x} + (|x| + |\dot{x}| - 1)\dot{x} + x|x| = 0$ has at least one periodic solution.

We can write the equations as

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$$

with $f(x, \dot{x}) = |x| + |\dot{x}| - 1$ and $g(x) = x|x|$.

We see that $f(x, y) = |x| + |y| - 1 > 0$ for $|x| + |y| > 1$, that is for $\sqrt{x^2 + y^2} > 1$. Further, $f(0, 0) = -1 < 0$, $g(0) = 0$, $g(x) = x|x| > 0$ for $x > 0$ and $g(x) = x|x| < 0$ for $x < 0$. We also have

$$\lim_{x\to\infty} G(x) = \lim_{x\to\infty} \int_0^x g(u)\mathrm{d}u = \lim_{x\to\infty} \mathrm{sgn}(x)\frac{x^3}{3} = \infty.$$

The conditions of theorem 11.2 are satisfied, so there exists at least one periodic solution.

11.10 Show that the origin is a centre for $\ddot{x} + (k\dot{x} + 1)\sin x = 0$.

We write the equation as $\ddot{x} + f(x)\dot{x} + g(x) = 0$ with $f(x) = k\sin x$ and $g(x) = \sin x$.

We see that f and g are odd functions, and f does not change sign for positive x in a neighborhood of the origin. Further, $g(x) > 0$ for $x > 0$ in a neighborhood of the origin.

Finally, we verify

$$g(x) = \sin(x) > \alpha k^2 \sin x(1 - \cos x)$$

for x small enough holds for an $\alpha > 1$ since the term $(1 - \cos x)$ can be made arbitrary small enough close to the origin. By theorem 11.3, the origin is a centre for the equation.

12.1 (ii) We are asked to find the bifurcation points of the system $\dot{x} = A(\lambda)x$ where

$$A(\lambda) = \begin{bmatrix} \lambda & 1-\lambda \\ 1 & \lambda \end{bmatrix}$$

We find the eigenvalues of A, μ_1 and μ_2 by solving

$$(\lambda - \mu)^2 - (1 - \lambda) = 0,$$

which has solution

$$\mu = \lambda \pm \sqrt{1 - \lambda}.$$

If $\lambda > 1$ we have complex conjugated eigenvalues, which give us an unstable spiral. If $\lambda < 1$ we have real eigenvalues. We check for which values of $0 < \lambda < 1$ gives positive eigenvalues. For $\mu_2 = \lambda - \sqrt{1 - \lambda}$ to be positive we need

$$\begin{aligned} &\lambda > \sqrt{1-\lambda} > 1 - \lambda, \\ &\lambda^2 > 1 - 2\lambda + \lambda^2, \\ &\lambda > \frac{1}{2}. \end{aligned}$$

Thus, for $\frac{1}{2} < \lambda < 1$ we have an unstable node. Similarly, we find that A has both a positive and a negative eigenvalue for $0 < \lambda < \frac{1}{2}$, which is a saddle point. Finally, for $\lambda < 0$, the eigenvalues have opposite sign, so we have a saddle point for $\lambda < 0$. We summarize in the following table.

$\lambda > 1$	Unstable spiral
$\frac{1}{2} < \lambda < 1$	Unstable node
$0 < \lambda < \frac{1}{2}$	Saddle
$\lambda < 0$	Saddle

The only bifurcation point is the saddle-node bifurcation, for $\lambda = \frac{1}{2}$.

12.9 The equilibrium points of the system

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y^2 - \lambda \end{aligned}$$

is given by $x = 0$ and $y^2 = \lambda$. We see that for negative λ, there are no equilibrium points. For $\lambda = 0$ there is one equilibrium points, at $(0, 0)$. If $\lambda > 0$ we have two equilibrium points, namely $(0, -\sqrt{\lambda}), (0, \sqrt{\lambda})$.

The matrix of linearization is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & \pm 2\sqrt{\lambda} \end{bmatrix}$$

at the points $(0, \pm\sqrt{\lambda})$. We see that $(0, \sqrt{\lambda})$ is an unstable node, while $(0, -\sqrt{\lambda})$ is a saddle point. Hence we have a saddle-node bifurcation and $\lambda = 0$ is a bifurcation point. When giving a sketch for different values of μ, we first see how the vector fields $(\dot{x}, \dot{y})$ varies in the four quadrants of the (x, y)-plane. See figure 2, 3 and 4 for a sketch of the phase diagram when $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$.

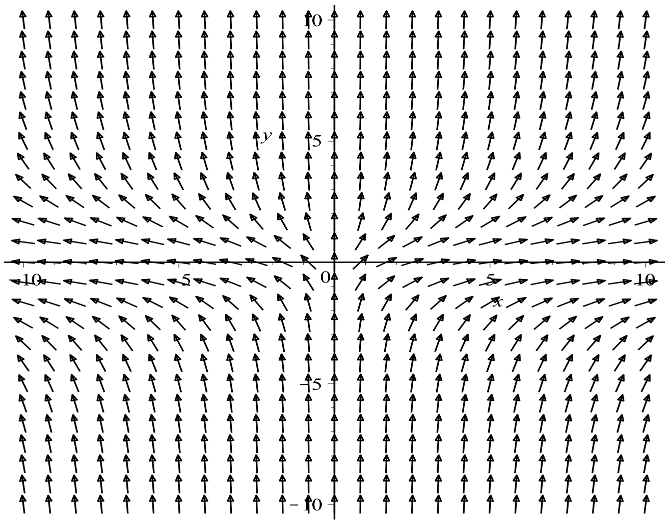

Figure 2: Phase daigram of $\dot{x} = x$, $\dot{y} = y^2 - \lambda$ when $\lambda = -1$.

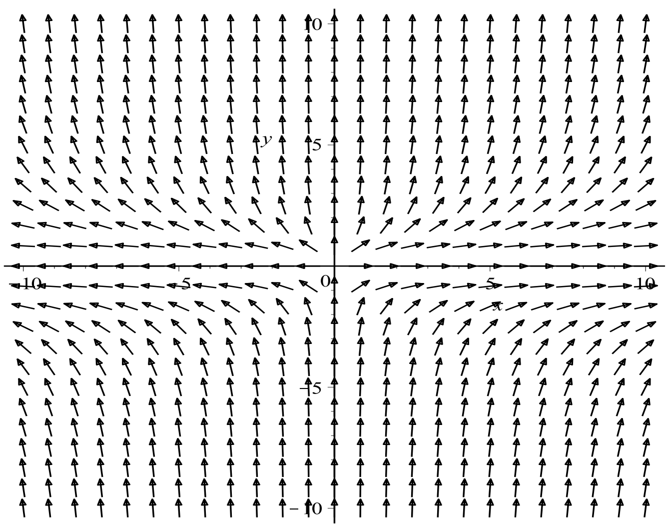

Figure 3: Phase daigram of $\dot{x} = x$, $\dot{y} = y^2 - \lambda$ when $\lambda = 0$.

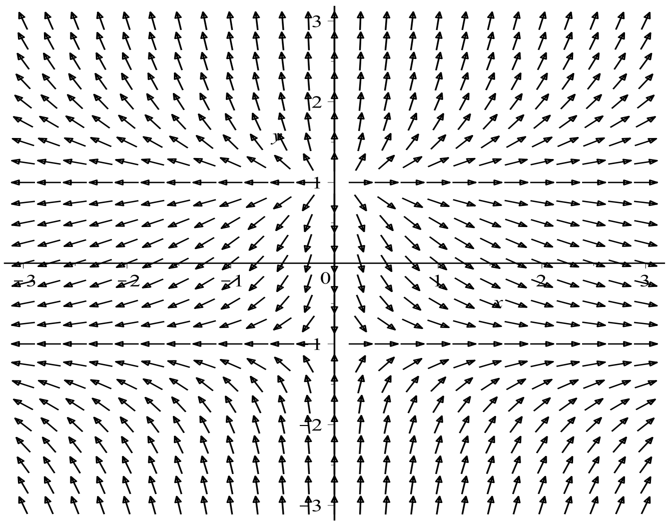

Figure 4: Phase daigram of $\dot{x} = x$, $\dot{y} = y^2 - \lambda$ when $\lambda = 1$.

12.19 For the system

$$\dot{x} = x(\mu - x)$$
$$\dot{y} = y(\mu - 2x)$$

we see that there are two equilibrium points, namely $(0, 0)$ and $(\mu, 0)$. The matrix of linearization is given by

$$\begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$$

at the point $(0, 0)$. Hence we see that for $\mu > 0$ we have an unstable node at the origin, while for $\mu < 0$ the origin is a stable node. Further, the matrix of linearization is

$$\begin{bmatrix} -\mu & 0 \\ 0 & -\mu \end{bmatrix}$$

at the point $(\mu, 0)$. Hence, we see that for $\mu > 0$, the point $(\mu, 0)$ is a stable node, while it is an unstable node for $\mu < 0$.

$\mu = 0$ is a transcritical bifurcation point, since then both eigenvalues of the system is zero. When drawing the phase diagram for different values of μ, it is a good idea to first give a sketch of how the vector field $(\dot{x}, \dot{y})$ varies when $x(\mu - x) = 0$ and $y(\mu - 2x) = 0$. See figure 5, 6 and 7 for a sketch of the phase diagram when $\mu = -1$, $\mu = 0$ and $\mu = 1$.

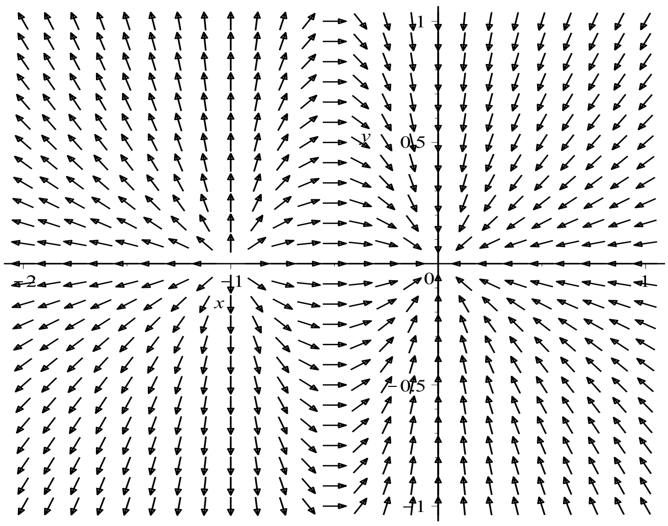

Figure 5: Phase daigram of $\dot{x} = x(\mu - x)$, $\dot{y} = y(\mu - 2x)$ when $\mu = -1$.

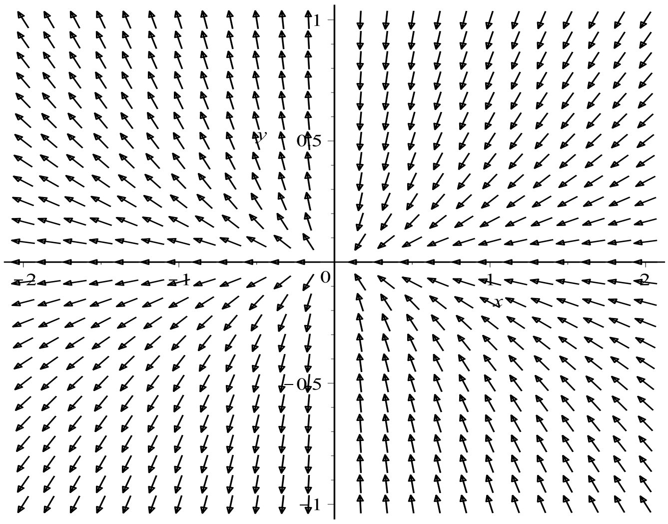

Figure 6: Phase daigram of $\dot{x} = x(\mu - x)$, $\dot{y} = y(\mu - 2x)$ when $\mu = 0$.

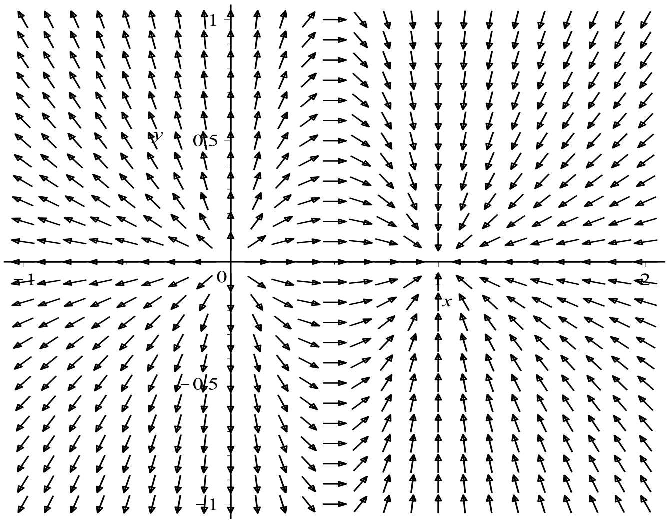

Figure 7: Phase daigram of $\dot{x} = x(\mu - x)$, $\dot{y} = y(\mu - 2x)$ when $\mu = 1$.

12.24 We see that the system

$$\begin{aligned}\dot{r} &= r(r^2 - \mu r + 1) \\ \dot{\theta} &= -1\end{aligned}$$

only has one equilibrium point, at the origin. Note that $\dot{r}$ changes sign if $r^2 - \mu r + 1$ changes sign. The solution of $r^2 - \mu r + 1 = 0$ is

$$r_{1,2} = \frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} - 1}$$

which is positive real if $|\mu| > 2$. We write

$$\dot{r} = r(r - r_1)(r - r_2)$$

where $r_1, r_2 > 0$. Note that $\dot{r} > 0$ if $r_1, r_2 < r$ and $\dot{r} < 0$ if $r_1 < r < r_2$. We see that the circle with radius r_1 centered at $(0, 0)$ is a stable limit cycle, while the circle with radius r_2 centered at $(0, 0)$ is an unstable limit cycle.

For $\mu < 2$ we see that there are no real solutions to $\dot{r} = 0$ and we only have to check one point to see that $\dot{r} > 0$ when $\mu < 2$, so we have an unstable spiral.