IDEA BEHIND HARTMAN-GROBMAN ($n=2$)

GIVEN $\dot{\vec{x}}=f(\vec{x})$ ST $\vec{x}=\vec{b}$ IS AN ISOLATED EP

- INTRODUCE A CHANGE OF VARIABLES $\vec{y}=\vec{x}-\vec{b}$

$$\Rightarrow \vec{y}'=\vec{x}'=f(\vec{x})=f(\vec{y}+\vec{b})$$

DEFINE $g(\vec{y})=f(\vec{y}+\vec{b}) \Rightarrow \vec{y}'=g(\vec{y})$ AND $\vec{y}=\vec{0}$ IS AN ISOLATED EP

$\Rightarrow$ WLOG WE CAN ASSUME $\vec{x}'=f(\vec{x})$ AND $\vec{x}=\vec{0}$ IS AN ISOLATED EP.

- $\exists \delta>0$ ST $f\in C^1(N_\delta(\vec{0})) \Rightarrow f(\vec{x})=f(\vec{0})+(f(\vec{x})-f(\vec{0}))=f(\vec{0})+\int_0^1 Df(\vec{0}+s\vec{x})\vec{x}\,ds$

$$=f(\vec{0})+\int_0^1 Df(\vec{0})\vec{x}\,ds+\int_0^1 (Df(\vec{0}+s\vec{x})-Df(\vec{0}))\vec{x}\,ds$$

$$=\theta+Df(\vec{0})\vec{x}+\int_0^1 \underbrace{(Df(\vec{0}+s\vec{x})-Df(\vec{0}))}_{\in C(N_\delta(\vec{0}))}\vec{x}\,ds$$

$$=Df(\vec{0})\vec{x}+o(|\vec{x}|)\quad \forall \vec{x}\in N_\delta(\vec{0}).$$

$\Rightarrow \vec{x}'=f(\vec{x})=A\vec{x}+o(|\vec{x}|)\ \forall\vec{x}\in N_\delta(\vec{0})$ WHERE $A=Df(\vec{0})$.

- LET $y=P^{-1}\vec{x}$ ... P INV MATRIX

$$\Rightarrow \vec{y}'=P^{-1}\vec{x}'=P^{-1}(A\vec{x}+o(|\vec{x}|))=P^{-1}A\vec{x}+P^{-1}o(|\vec{x}|)=P^{-1}AP\vec{y}+P^{-1}o(P\vec{y})=J\vec{y}+o(|\vec{y}|)$$

$\Rightarrow$ WLOG WE CAN ASSUME $\exists\delta>0$ ST $\vec{x}'=J\vec{x}+o(|\vec{x}|)\ \forall\vec{x}\in N_\delta(\vec{0})$.

- CASE 1: $J=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ WITH $\lambda_1<0<\lambda_2$.

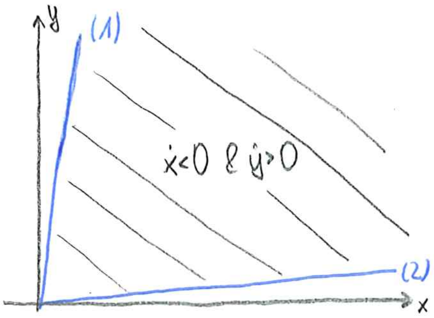
PICK $\vec{x}(0)=\begin{pmatrix}x_0\\ y_0\end{pmatrix}$ ST $x_0>0$ AND $y_0>0$. WHAT CAN WE SAY ABOUT THE PHASE PATH THROUGH $\vec{x}(0)$?

$\vec{x}'=J\vec{x}+o(|\vec{x}|) \Rightarrow \forall\varepsilon>0\ \exists\eta(\varepsilon)>0$ ST $o(|\vec{x}|)\le\varepsilon(x+y)\ \forall\vec{x}\in N_{\eta(\varepsilon)}(\vec{0})\subseteq N_\delta(\vec{0})$ WITH $x,y>0$

CHOOSE $\varepsilon>0$ ST $\lambda_1+2\varepsilon<0<\lambda_2-2\varepsilon$ AND $\vec{x}\in N_{\eta(\varepsilon)}(\vec{0})$ ST $x,y>0$

$$\Rightarrow (\lambda_1-\varepsilon)x-\varepsilon y\le x'\le(\lambda_1+\varepsilon)x+\varepsilon y \quad\rightsquigarrow\quad x'\le 0 \text{ IF } (\lambda_1+\varepsilon)x+\varepsilon y\le 0 \rightsquigarrow y\le-\frac{(\lambda_1+\varepsilon)}{\varepsilon}x$$

$$-\varepsilon x+(\lambda_2-\varepsilon)y\le y'\le\varepsilon x+(\lambda_2+\varepsilon)y \quad\rightsquigarrow\quad 0\le y' \text{ IF } 0\le-\varepsilon x+(\lambda_2-\varepsilon)y \rightsquigarrow \frac{\varepsilon}{(\lambda_2-\varepsilon)}x\le y$$



(1)… $y=-\frac{(\lambda_1+\varepsilon)}{\varepsilon}x$

(2)… $y=\frac{\varepsilon}{(\lambda_2-\varepsilon)}x$

INSIDE \\\ $\frac{1}{(\lambda_1+\varepsilon)x+\varepsilon y} \leq \frac{1}{x'} \leq \frac{1}{(\lambda_1-\varepsilon)x-\varepsilon y}$

$-\varepsilon x+(\lambda_2-\varepsilon)y \leq y' \leq \varepsilon x+(\lambda_2+\varepsilon)y$

$\Rightarrow \frac{-\varepsilon x+(\lambda_2-\varepsilon)y}{(\lambda_1+\varepsilon)x+\varepsilon y} \leq \frac{y'}{x'} \leq \frac{\varepsilon x+(\lambda_2+\varepsilon)y}{(\lambda_1-\varepsilon)x-\varepsilon y}$

PHASE PATH TO

$\vec{x}' = B_1\vec{x}$ (*)

$B_1=\begin{pmatrix}\lambda_1+\varepsilon & \varepsilon \\ -\varepsilon & \lambda_2-\varepsilon\end{pmatrix}$

PHASE PATH TO

$\vec{x}' = B_2\vec{x}$ (**)

$B_2=\begin{pmatrix}\lambda_1-\varepsilon & -\varepsilon \\ \varepsilon & \lambda_2+\varepsilon\end{pmatrix}$

(*) EIGENVALUES: $\mu_{1,2}$

$(\lambda_1+\varepsilon-\mu)(\lambda_2-\varepsilon-\mu)+\varepsilon^2=0$

$\mu^2-(\lambda_1+\lambda_2)+\varepsilon^2+(\lambda_1+\varepsilon)(\lambda_2-\varepsilon)^2$

$\mu_{12}=\frac{\lambda_1+\lambda_2}{2}\pm\sqrt{\frac{(\lambda_1+\lambda_2)^2}{4}-\lambda_1\lambda_2-(\lambda_2-\lambda_1)\varepsilon}$

$=\frac{\lambda_1+\lambda_2}{2}\pm\sqrt{\frac{(\lambda_1-\lambda_2)^2}{4}-(\lambda_2-\lambda_1)\varepsilon+\varepsilon^2-\varepsilon^2}$

$=\frac{\lambda_1+\lambda_2}{2}\pm\left(\frac{\lambda_2-\lambda_1}{2}-\varepsilon\right)+o(\varepsilon)$

$\Rightarrow \mu_1=\lambda_1+\varepsilon+o(\varepsilon)$ & $\mu_2=\lambda_2-\varepsilon+o(\varepsilon)$

EIGENVECTORS:

μ_1: $(B_1-\mu_1 I)\vec{x}=0 \Rightarrow \begin{pmatrix}o(\varepsilon) & \varepsilon \\ -\varepsilon & \lambda_2-\lambda_1-2\varepsilon+o(\varepsilon)\end{pmatrix}\vec{x}=\vec{0}$

$\Rightarrow y=\frac{\varepsilon}{\lambda_2-\lambda_1-2\varepsilon+o(\varepsilon)}x$… CLOSE TO THE x-AXIS

μ_2: $(B_1-\mu_2 I)\vec{x}=\vec{0} \Rightarrow \begin{pmatrix}\lambda_1-\lambda_2+2\varepsilon+o(\varepsilon) & \varepsilon \\ -\varepsilon & o(\varepsilon)\end{pmatrix}\vec{x}=\vec{0}$

$y=\frac{\lambda_2-\lambda_1-2\varepsilon+o(\varepsilon)}{\varepsilon}x$… CLOSE TO THE y-AXIS

(**) EIGENVALUES $\mu_{1,2}$

$(\lambda_1-\varepsilon-\mu)(\lambda_2+\varepsilon-\mu)+\varepsilon^2=0$

$\mu^2-(\lambda_1+\lambda_2)\mu+\varepsilon^2+(\lambda_1-\varepsilon)(\lambda_2+\varepsilon)=0$

$\mu_{12}=\frac{\lambda_1+\lambda_2}{2}\pm\sqrt{\frac{(\lambda_1+\lambda_2)^2}{4}-\lambda_1\lambda_2-(\lambda_1-\lambda_2)\varepsilon}$

$=\frac{\lambda_1+\lambda_2}{2}\pm\left(\frac{\lambda_1-\lambda_2}{2}-\varepsilon\right)+o(\varepsilon)$

$\Rightarrow \mu_1=\lambda_1-\varepsilon+o(\varepsilon)$ & $\mu_2=\lambda_2+\varepsilon+o(\varepsilon)$

EIGENVECTORS:

μ_1: $(B_2-\mu_1 I)\vec{x}=\begin{pmatrix}o(\varepsilon) & -\varepsilon \\ \varepsilon & \lambda_2-\lambda_1+2\varepsilon+o(\varepsilon)\end{pmatrix}\vec{x}=\vec{0}$

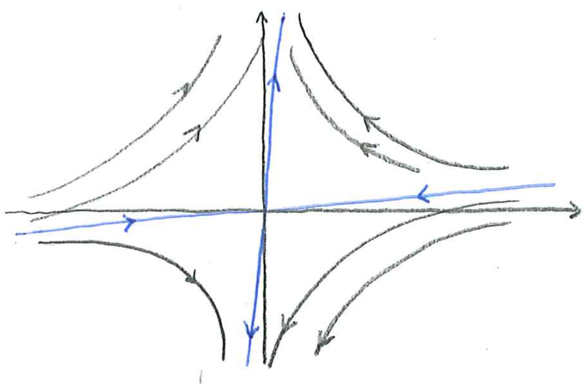
$\Rightarrow y=-\frac{\varepsilon}{\lambda_2-\lambda_1+2\varepsilon+o(\varepsilon)}x$

μ_2: $(B_2-\mu_2 I)\vec{x}=\begin{pmatrix}\lambda_1-\lambda_2-2\varepsilon+o(\varepsilon) & -\varepsilon \\ \varepsilon & o(\varepsilon)\end{pmatrix}$

$y=-\frac{\lambda_2-\lambda_1+2\varepsilon+o(\varepsilon)}{\varepsilon}x$

PHASE PORTRAITS

(*)



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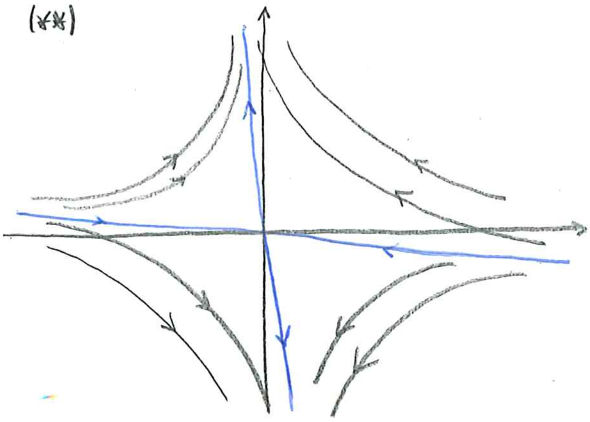
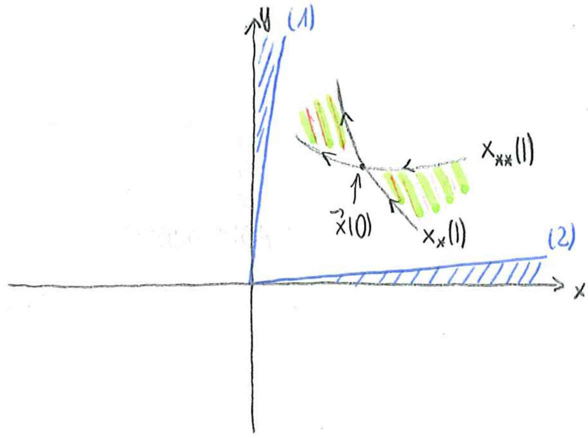


⇒



⇒ $\vec{x}(t)$ MUST LIE INSIDE ///

$x_*(t)$.. SOLUTION TO (*)

$x_{**}(t)$.. SOLUTION TO (**)

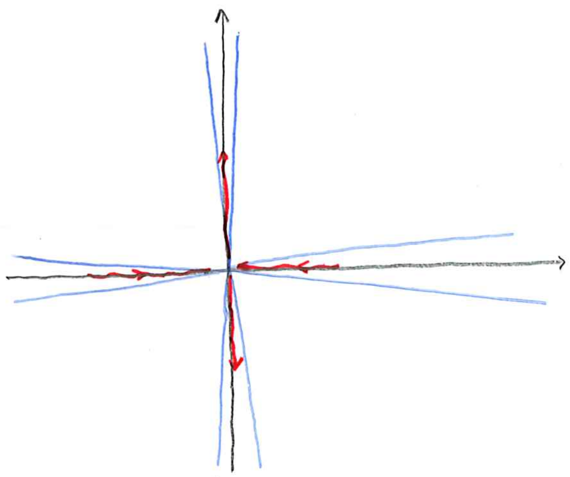
⇒ SOLUTION CAN'T TEND TO $\vec{0}$ WITHOUT ENTERING ///

$\vec{x}(t)$ CAN ONLY ENTER /// BOUNDED BY (1) FORWARD IN TIME.

/// BOUNDED BY (2) BACKWARD IN TIME.

FURTHERMORE (1) AND (2) DEPEND ON ε AND GET CLOSER AND CLOSER TO THE y-AXIS AND x-AXIS RESPECTIVELY AS $\varepsilon \to 0$

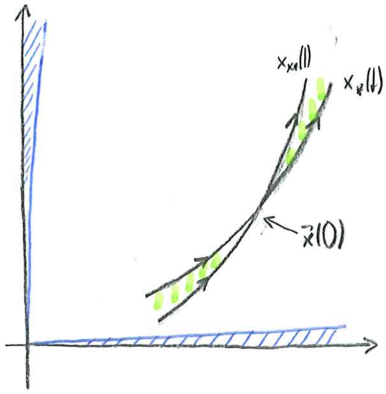
⇒



- ALL PHASE PATHS EXCEPT ONE IN EACH $\angle$ WILL END UP OUTSIDE $\angle$ AS $\varepsilon \to 0$
- $\exists!$ SOLUTION IN EACH HORIZONTAL $\angle$ WHICH TENDS TO $\vec{0}$ AS $t \to \infty$
  ~ STABLE MANIFOLD
- $\exists!$ SOLUTION IN EACH VERTICAL $\vee$ WHICH TENDS TO $\vec{0}$ AS $t \to -\infty$
  ~ UNSTABLE MANIFOLD

-) CASE 2 $\quad J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ WITH $0 < \lambda_1 < \lambda_2$

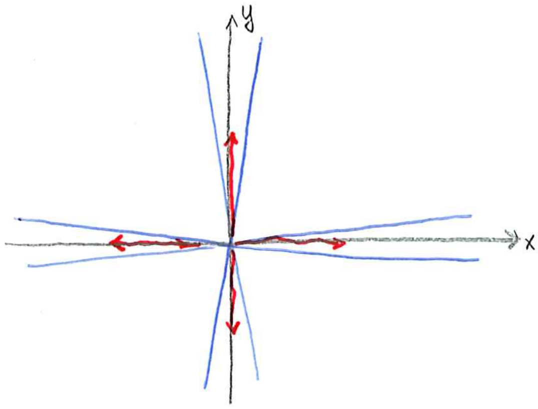
FOLLOWING THE SAME LINES YIELDS



$\Rightarrow \vec{x}(t)$ MUST LIE INSIDE ///

///

$\Rightarrow$



EXCEPT

-) ALL PHASE PATHS $\vee$ ONE IN EACH $\angle$ WILL END UP OUTSIDE $\angle$ AS $\varepsilon \to 0$

-) THE $\rightsquigarrow$ PLAY THE ROLE OF THE SPAN OF THE EIGENVECTORS FOR A LINEAR SYSTEM

-) ALL PHASE PATHS SATISFY $\vec{x}(t) \to 0$ AS $t \to -\infty$ FOR $\vec{x}(0)$ CLOSE ENOUGH TO $\vec{0}$.

-) CASE 3: $\quad J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\dot{x} = -x - \frac{y}{\ln(\sqrt{x^2+y^2})}$$
$$\dot{y} = -y + \frac{x}{\ln(\sqrt{x^2+y^2})}$$

$\to f \in C^1(\mathbb{R}^2)$ $(f(\vec{0}) = \vec{0})$

WITH LINEARIZATION $\dot{\vec{x}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \vec{x}$ AT THE EP $\vec{x} = \vec{0}$

$\Rightarrow$ THE LIN SYSTEM HAS AS PHASE PATHS STRAIGHT LINES THROUGH $\vec{0}$

$\Rightarrow \vec{0}$ ... NODE

POLAR COORDINATES $\quad r^2 = x^2 + y^2 \Rightarrow (r^2(t))' = 2x\dot{x}(t) + 2y\dot{y}(t) = -2x^2(t) - 2y^2(t) = -2r^2(t)$

$\Rightarrow r^2(t) = e^{-2t} r^2(0)$ .... FOR $r(0) > 0$

$$\theta = \arctan\left(\frac{y}{x}\right) \Rightarrow \theta'(t) = \frac{1}{1 + \frac{y^2(t)}{x^2(t)}} \frac{y'(t)x(t) - y(t)x'(t)}{x^2(t)} = \frac{y'(t)x(t) - y(t)\dot{x}(t)}{r^2(t)} = +\frac{1}{\ln(r(t))} = +\frac{1}{-t + \ln(r(0))}$$

$$\Rightarrow \theta(t) = \theta(0) - \ln|-t + \ln(r(0))| + \ln|\ln(r(0))| = \theta(0) - \ln\left|1 - \frac{t}{\ln(r(0))}\right|$$

$\Rightarrow r(t) \to 0 \quad (t\to\infty)$ AND $\theta(t) \to -\infty \quad (t \to \infty)$

$\Rightarrow$ SOLUTIONS BEHAVE LIKE A STABLE SPIRAL / FOCUS SPIRALING $\circlearrowright$

REASON $\lambda_1 = \lambda_2 = \lambda$ ~ MIGHT YIELD FOR THE EIGENVALUES OF THE APPR. EITHER

$\mu_1 = \overline{\mu_2}$ OR μ_1 AND μ_2 REAL.

•) CASE 4:

$$J = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad a<0,\ b>0$$

PICK $\vec{x}(0) = \binom{x_0}{y_0}$ ST $x_0>0$ AND $y_0>0$. WHAT CAN WE SAY ABOUT THE PHASE PATH THROUGH $\vec{x}(0)$. ?

$\vec{x}' = J\vec{x} + o(|\vec{x}|) \Rightarrow \forall \varepsilon > 0\ \exists \eta(\varepsilon)>0$ ST $o(|\vec{x}|) \le \varepsilon(x+y)$ $\forall \vec{x} \in N_{\eta(\varepsilon)}(\vec{0}) \subseteq N_\delta(\vec{0})$ AND $x,y>0$

CHOOSE $\varepsilon>0$ ST $a+2\varepsilon<0$ AND $b-\varepsilon>0$ AND $\vec{x} \in N_{\eta(\varepsilon)}(\vec{0})$ WITH $x,y>0$

$$\Rightarrow (a-\varepsilon)x + (b-\varepsilon)y \le \dot{x} \le (a+\varepsilon)x + (b+\varepsilon)y$$

$$(-b-\varepsilon)x + (a-\varepsilon)y \le \dot{y} \le (-b+\varepsilon)x + (a+\varepsilon)y$$

POLAR COORDINATES:

$r^2(t) = x^2(t) + y^2(t) \Rightarrow (r^2(t))' = 2x\dot{x}(t) + 2y\dot{y}(t)$

$\Rightarrow$ •) $2(a-\varepsilon)x^2 + 2(b-\varepsilon)xy + (-b-\varepsilon)2xy + (a-\varepsilon)2y^2 \le (r^2)'$

$\rightsquigarrow 2(a-\varepsilon)r^2 - 2\varepsilon r^2 \le (r^2)'$

$\rightsquigarrow 2(a-2\varepsilon)r^2 \le (r^2)'$

•) $(r^2)' \le 2(a+\varepsilon)x^2 + 2(b+\varepsilon)xy + 2(-b+\varepsilon)xy + 2(a+\varepsilon)y^2$

$\rightsquigarrow (r^2)' \le 2(a+\varepsilon)r^2 + 2\varepsilon r^2$

$\rightsquigarrow (r^2)' \le 2(a+2\varepsilon)r^2 < 0$ IF $r>0$

$\Rightarrow$ THE RADIUS $r(t)$ IS STRICTLY DECREASING

$\theta(t) = \arctan\left(\frac{y(t)}{x(t)}\right) \rightsquigarrow \theta'(t) = \frac{y'(t)x(t) - y(t)x'(t)}{r^2(t)}$

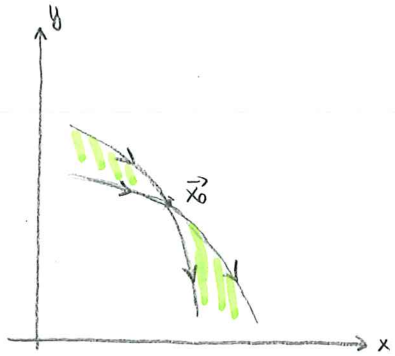
$\Rightarrow$ •) $\frac{(-b-\varepsilon)x^2 + (a-\varepsilon)xy - (a+\varepsilon)xy - (b+\varepsilon)y^2}{r^2} = \le \theta'$

$\rightsquigarrow \frac{-2br^2 - 2\varepsilon xy}{r^2} \le -2(b+\varepsilon) \le \theta'$

•) $\theta' \le \frac{(-b+\varepsilon)x^2 + (a+\varepsilon)xy - (a-\varepsilon)xy - (b+\varepsilon)y^2}{r^2}$

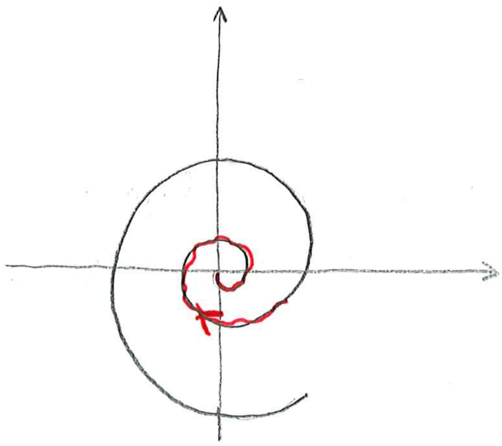
$\rightsquigarrow \theta' \le -2(b-\varepsilon)$

THE ∡ θ IS STRICTLY DECREASING $\Rightarrow$ ↻



$\Rightarrow \vec{x}(t)$ MUST LIE INSIDE

$\Rightarrow |\vec{x}(t)|$ IS DECREASING



ALL SOLUTIONS STARTING CLOSE ENOUGH TO $\vec{0}$ WILL SPIRAL TOWARDS $\vec{0}$

•) CASE 5:

$$J = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

$$\dot{x} = -y + x\sqrt{x^2+y^2}$$
$$\dot{y} = x + y\sqrt{x^2+y^2}$$
$\leadsto f \in C^1(\mathbb{R})$

WITH LINEARISATION $\dot{\vec{x}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{x}$ AT THE EP $\vec{x} = \vec{0}$

$\Rightarrow$ THE LIN SYSTEM HAS AS PHASE PATHS CIRCLE SURROUNDING $\vec{0}$

$\Rightarrow \vec{0}$ ... CENTRE

POLAR COORDINATES:

$$r^2 = x^2+y^2 \to (r^2(t))' = 2x\dot{x}(t) + 2y\dot{y}(t) = +r^{5/2}(t)$$ $\Rightarrow$ RADIUS IS INCREASING

$$\theta'(t) = \frac{y'x(t) - y x'(t)}{r^2(t)} = 1$$

$\Rightarrow$ SOLUTIONS BEHAVE LIKE AN UNSTABLE FOCUS SPIRALING ↺

REASON: $\lambda_1 = \bar{\lambda}_2 = ib \leadsto$ MIGHT YIELD FOR THE EIGENVALUES OF THE APPR $\mu_1 = \bar{\mu}_2 = a + ib$ $(a \neq 0)$