THM: LET $E \subseteq \mathbb{R}^n$ OPEN AND $f: E \to \mathbb{R}^n$ LOCALLY LIPSCHITZ CONTINUOUS. ASSUME THAT THE IVP $\dot{\vec{x}} = f(\vec{x}), \vec{x}(0) = \vec{x}_0$ HAS A UNIQUE SOL $\vec{x}(t, \vec{x}_0)$ DEFINED ON $[0, b]$. THEN THERE EXISTS $\delta > 0$ AND $K > 0$ ST $\forall \vec{y} \in N_\delta(\vec{x}_0)$ THE IVP $\dot{\vec{x}} = f(\vec{x}), \vec{x}(0) = \vec{y}$ HAS A UNIQUE SOL $\vec{x}(t, \vec{y})$ DEFINED ON $[0, b]$ WHICH SATISFIES

$$|\vec{x}(t, \vec{y}) - \vec{x}(t, \vec{x}_0)| \leq e^{K|t|} |\vec{x}_0 - \vec{y}| \quad \& \quad \lim_{\vec{y} \to \vec{x}_0} \vec{x}(t, \vec{y}) = \vec{x}(t, \vec{x}_0) \text{ UNIF } \forall t \in [0, b].$$

PROOF. $\vec{x}(t, \vec{x}_0)$ DEFINED ON $[0, b] \Rightarrow \exists \varepsilon > 0$ ST $\bar{\delta}(\vec{x}(t, \vec{x}_0)) \geq \varepsilon \quad \forall t \in [0, b]$

$$\left(\bar{\delta}(\vec{x}) = \min\left(\operatorname{dist}(x, \partial E), \frac{1}{|\vec{x}|} \right) \right)$$

LET $\vec{y} \in N_\delta(\vec{x}_0) \Rightarrow$ ∙) $|\vec{y}| \leq |\vec{y} - \vec{x}_0| + |\vec{x}_0| \leq \delta + \frac{1}{\varepsilon}$

∙) $\varepsilon \leq \operatorname{dist}(\vec{x}_0, \partial E) \leq \operatorname{dist}(\vec{y}, \partial E) + |\vec{y} - \vec{x}_0| \leq \operatorname{dist}(\vec{y}, \partial E) + \delta$

$$\Rightarrow \varepsilon - \delta \leq \operatorname{dist}(\vec{y}, \partial E)$$

$\Rightarrow$ CHOOSE $\delta < \frac{1}{2} \min(\varepsilon, \frac{1}{\varepsilon})$ THEN $\vec{y} \in E$ WITH $\bar{\delta}(\vec{y}) \geq \frac{1}{2}\varepsilon$

$\Rightarrow \exists$ A CLOSED INTERVAL $I_y \subseteq [0, b]$ ST $\vec{x}(t, \vec{y})$ IS THE UNIQUE SOL ON I_y TO THE IVP $\dot{\vec{x}} = f(\vec{x}), \vec{x}(0) = \vec{y}$. AND $\bar{\delta}(\vec{x}(t, \vec{y})) \geq \frac{\varepsilon}{4} \quad \forall t \in I_{\vec{y}}$

CLAIM: $|\vec{x}(t, \vec{y}) - \vec{x}(t, \vec{x}_0)| \leq e^{L|t|} |\vec{x}_0 - \vec{y}| \quad \forall t \in I_y$

$\{\vec{x} \in E \mid \bar{\delta}(\vec{x}) \geq \frac{\varepsilon}{4}\}$ IS A COMPACT SUBSET OF E

$\overset{\text{LEM 2}}{\Rightarrow} f$ IS LIPSCHITZ CONTINUOUS ON $\{\vec{x} \in E \mid \bar{\delta}(\vec{x}) \geq \frac{3}{4}\}$ WITH LIPSCHITZ CONSTANT L

$\Rightarrow$ FOR $t \in I_y$ WITH $t \geq 0$

$$|\vec{x}(t, \vec{y}) - \vec{x}(t, \vec{x}_0)| \leq |\vec{y} - \vec{x}_0| + \int_0^t |f(\vec{x}(s, \vec{y})) - f(\vec{x}(s, \vec{x}_0))| \, ds$$

$$\leq |\vec{y} - \vec{x}_0| + L \int_0^t |\vec{x}(s, \vec{y}) - \vec{x}(s, \vec{x}_0)| \, ds$$

$\Rightarrow$ (GRONWALL) $\quad |\vec{x}(t, \vec{y}) - \vec{x}(t, \vec{x}_0)| \leq e^{Lt} |\vec{x}_0 - \vec{y}|.$

$\Rightarrow$ ∙) $|\vec{x}(t, \vec{y})| \leq |\vec{x}(t, \vec{y}) - \vec{x}(t, \vec{x}_0)| + |\vec{x}(t, \vec{x}_0)| \leq e^{Lt} |\vec{x}_0 - \vec{y}| + \frac{1}{\varepsilon}$

∙) $\varepsilon \leq \operatorname{dist}(\vec{x}(t, \vec{x}_0), \partial E) \leq \operatorname{dist}(\vec{x}(t, \vec{y}), \partial E) + |\vec{x}(t, \vec{y}) - \vec{x}(t, \vec{x}_0)| \leq \operatorname{dist}(\vec{x}(t, \vec{y}), \partial E) + e^{Lt} |\vec{x}_0 - \vec{y}|$

$I_y = [c,d] \Rightarrow \vec{x}(t,\vec{y}) \in \{\vec{x} \in E \mid \bar{\delta}(\vec{x}) \geq \frac{\varepsilon}{4}\}$ $\forall t \in [0,d]$ IF

$$\frac{1}{\varepsilon} + e^{Ld} |\vec{x}_0 - \vec{y}| < \frac{4}{\varepsilon} \quad \text{AND} \quad \varepsilon - e^{Ld} |\vec{x}_0 - \vec{y}| \geq \frac{\varepsilon}{4} \qquad (*)$$

$\Rightarrow$ IF $|\vec{x}_0 - \vec{y}|$ IS SMALL ENOUGH $(*)$ CAN BE SATISFIED FOR $d = b$

$\Rightarrow$ (L INDEPENDENT OF $\vec{y}$)

$$\exists \delta > 0 \text{ ST } |\vec{x}(t,\vec{x}_0) - \vec{x}(t,\vec{y})| \leq e^{Lt} |\vec{y} - \vec{x}_0| \quad \forall \vec{y} \in N_\delta(\vec{x}_0) \text{ AND } \forall\, 0 \leq t \leq b$$

FOR $t < 0$ WE CAN FOLLOW THE SAME LINES, SINCE FOR $t \leq 0$ AND $\vec{z}(t,\vec{y}) = \vec{x}(-t,\vec{y})$

$$\vec{z}(-t,\vec{y}) - \vec{z}(-t,\vec{x}_0) = \vec{x}(t,\vec{y}) - \vec{x}(t,\vec{x}_0) = \vec{y} - \vec{x}_0 + \int_0^t f(\vec{x}(s,y)) - f(\vec{x}(s,\vec{x}_0))\, ds$$

$$= \vec{y} - \vec{x}_0 + \int_0^t f(\vec{z}(-s,y)) - f(\vec{z}(-s,\vec{x}_0))\, ds$$

$$= \vec{y} - \vec{x}_0 + \int_0^{-t} f(\vec{z}(s,\vec{x}_0)) - f(\vec{z}(s,\vec{y}))\, ds.$$

$\Rightarrow \exists \delta > 0$ ST $|\vec{x}(t,\vec{x}_0) - \vec{x}(t,\vec{y})| \leq e^{L|t|} |\vec{y} - \vec{x}_0|$ $\forall \vec{y} \in N_\delta(\vec{x}_0)$ AND $\forall t \in [a,b]$

$\Rightarrow \lim_{\vec{y} \to \vec{x}_0} \vec{x}(t,\vec{y}) = \vec{x}(t,\vec{x}_0)$ UNIFORMLY $\forall t \in [a,b]$.

THM: LET $E_1 \times E_2 \subseteq \mathbb{R}^n \times \mathbb{R}^m$ OPEN AND $f: E_1 \times E_2 \to \mathbb{R}^n$ LOCALLY LIPSCHITZ CONTINUOUS. ASSUME THAT THE IVP $\dot{\vec{x}} = f(\vec{x}, \vec{\mu}_0)$ $\vec{x}(0) = \vec{x}_0$ HAS A UNIQUE SOL $\vec{x}(t,\vec{x}_0,\vec{\mu}_0)$ DEFINED ON $[a,b]$, THEN THERE EXISTS $\delta > 0$ AND $K > 0$ ST $\forall \vec{\lambda} \in N_\delta(\vec{\mu}_0)$ AND $\forall \vec{y} \in N_\delta(\vec{x}_0)$ THE IVP $\dot{\vec{x}} = f(\vec{x},\vec{\lambda}), \vec{x}(0) = \vec{y}$ HAS A UNIQUE SOL $\vec{x}(t,\vec{y},\vec{\lambda})$ DEFINED ON $[a,b]$ WHICH SATISFIES

$$|\vec{x}(t,\vec{y},\vec{\lambda}) - \vec{x}(t,\vec{x}_0,\vec{\mu}_0)| \leq e^{K|t|}(|\vec{y} - \vec{x}_0| + |\vec{\lambda} - \vec{\mu}_0|) \quad \& \quad \lim_{(\vec{y},\vec{\lambda}) \to (\vec{x}_0,\vec{\mu}_0)} \vec{x}(t,\vec{y},\vec{\lambda}) = \vec{x}(t,\vec{x}_0,\vec{\mu}_0) \text{ UNIF } \forall t \in [a,b].$$

PROOF: LET $\vec{v} = \begin{pmatrix} \vec{y} \\ \vec{\lambda} \end{pmatrix}$ AND $\vec{z}(t,\vec{v}) = \begin{pmatrix} \vec{x}(t,\vec{v}) \\ \vec{\lambda} \end{pmatrix} = \begin{pmatrix} \vec{x}(t,\vec{y},\vec{\lambda}) \\ \vec{\lambda} \end{pmatrix}$.

$$\Rightarrow \dot{\vec{z}}(t,\vec{v}) = \begin{pmatrix} \dot{\vec{x}}(t,\vec{y},\vec{\lambda}) \\ 0 \end{pmatrix} = \begin{pmatrix} f(\vec{x}(t,\vec{y},\vec{\lambda}),\vec{\lambda}) \\ 0 \end{pmatrix} = \begin{pmatrix} f(\vec{z}(t,\vec{v})) \\ 0 \end{pmatrix} = g(\vec{z}(t,\vec{v})) \text{ AND } \vec{z}(0,\vec{v}) = \vec{v}$$

$\Rightarrow$ CAN APPLY THE LAST THM TO $\vec{z}(t,\vec{v}_0) = \begin{pmatrix} \vec{x}(t,\vec{x}_0,\vec{\mu}_0) \\ \vec{\mu}_0 \end{pmatrix}$

$\Rightarrow \exists \delta > 0$ AND $K > 0$ ST $\forall \vec{v} \in N_{2\delta}(\vec{v}_0)$ THE IVP $\dot{\vec{z}} = g(\vec{z}), \vec{z}(0) = \vec{v}$ HAS A UNIQUE SOL $\vec{z}(t,\vec{v})$ DEFINED ON $[a,b]$ WHICH SATISFIES

$$|\vec{x}(t,\vec{y},\vec{\lambda})-\vec{x}(t,\vec{x}_0,\vec{\mu}_0)| \le |\vec{z}(t,\vec{v})-\vec{z}(t,\vec{w}_0)| \le e^{K|t|}|\vec{v}-\vec{w}| \le e^{K|t|}(|\vec{y}-\vec{x}_0|+|\vec{\lambda}-\vec{\mu}|) \quad \forall t\in[a,b].$$

$\Rightarrow$ $\exists\, \delta>0$ AND $K>0$ ST $\forall \vec{y}\in N_\delta(\vec{x}_0)$ AND $\forall \vec{\lambda}\in N_\delta(\vec{\mu}_0)$ THE IVP $\dot{\vec{x}}=f(\vec{x},\vec{\lambda}),\ \vec{x}(0)=\vec{y}$ HAS A UNIQUE SOL $\vec{x}(t,\vec{y},\vec{\lambda})$ DEFINED ON $[a,b]$ WHICH SATISFIES

SINCE $$|\vec{x}(t,\vec{y},\vec{\lambda})-\vec{x}(t,\vec{x}_0,\vec{\mu}_0)| \le e^{K|t|}(|\vec{y}-\vec{x}_0|+|\vec{\lambda}-\vec{x}_0|) \quad \forall t\in[a,b].$$

$$\Rightarrow \lim_{(\vec{y},\vec{\lambda})\to(\vec{x}_0,\vec{\mu}_0)} \vec{x}(t,\vec{y},\vec{\lambda}) = \vec{x}(t,\vec{x}_0,\vec{\mu}_0) \text{ UNIF } \forall t\in[a,b].$$