75045 Dynamical systems 19. May 1998

Suggested solution

Problem 1

The Sierpinski tetrahedron is the fractal set associated with a set of four similitudes, namely the maps $x \mapsto \frac{1}{2}(x+c)$ where x is one of the corners of the given tetrahedron. The standard formula for the fractal dimension D gives us

$$4\Big(\frac{1}{2}\Big)^D = 1$$

which has the solution[1] $D = 2$.

Problem 2

(This is very standard. See Perko chapter 2. I excpect to see the mention of saddles, foci, nodes, and centers, with foci and nodes classified into stable and unstable ones.)

Problem 3

a To find equilibrium points, set $\dot{x} = 0$ to get $x^2 = y^2$, and $\dot{y} = 0$, which in conjunction with what we just found makes $x^2 = y^2 = 1/2$. So the equilibrium points are

$$\Big(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2} \Big)$$

with all four combinations of signs allowed.

Writing $f(x,y)$ for the righthand side of the equations, we get for the derivative of f

$$\mathrm{D}f(x,y) = \begin{bmatrix} 2x & -2y \\ -2x & -2y \end{bmatrix}.$$

This leads to the following classification of fixed points.

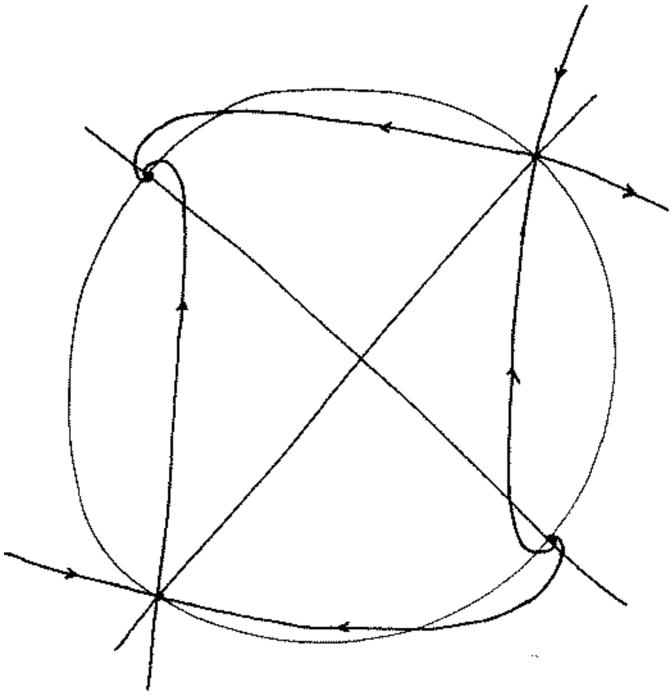
(x,y)	$\mathrm{D}f(x,y)$	eigenvalues	classification
$\frac{\sqrt{2}}{2}(\ \ 1,\ \ 1)$	$\sqrt{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$	± 2	saddle
$\frac{\sqrt{2}}{2}(\ \ 1,-1)$	$\sqrt{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$	$\sqrt{2}(\ \ 1 \pm i)$	unstable focus
$\frac{\sqrt{2}}{2}(-1,\ \ 1)$	$\sqrt{2}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$	$\sqrt{2}(-1 \pm i)$	stable focus
$\frac{\sqrt{2}}{2}(-1,-1)$	$\sqrt{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	± 2	saddle

[1] This example clearly contradicts the "definition" one sometimes sees of a fractal being a set whose fractal dimension is not a integer.

The above is the linear classification of the equilibrium points. However, since the system is C^2 and all the equilibria are hyperbolic, the same classification holds nonlinearly.[2]

The vector field has index 1 with respect to any cycle. Since saddle points have index -1 and foci have index 1, the sum of the indexes of all equilibria is 0. Hence no cycle can surround them all.

b



The unit circle and the two lines $y = \pm x$ are not part of the phase diagram, but are included for reference. It is helpful to note that the vector field is vertical ($\dot{x} = 0$) precisely on the lines $y = \pm x$, while it is horizontal ($\dot{y} = 0$) precisely on the unit circle. The arrows are not too visible in the scanned in version of the picture, but their directions should be clear from the fact that the south-east focus is unstable, the north-west one is stable, and the other two equilibria are saddle points.

Problem 4

The linearized system is $\dot{x} = y$, $\dot{y} = 0$. The matrix of this system has only the eigenvalue 0, so linearized analysis does not settle the question of stability. We look for a Liapunov function for the system. We want to try a low order polynomial in (x, y) involving only terms of even degree, and compute a few derivatives:

$$(x^2)' = 2x\dot{x} = 4xy - 2x^4 \qquad (x^4)' = 4x^3\dot{x} = 8x^3y - 4x^6$$
$$(y^2)' = 2y\dot{y} = -8x^3y \qquad (y^4)' = 4y^3\dot{y} = -16x^3y^3$$

and looking at this we realize that $V(x, y) = x^4 + y^2$ satisfies $\dot{V} = -4x^6 \le 0$, so at least the origin is stable.[3]

[2] C^1 is enough for the saddle points.

[3] If this had failed, we might have tried to involve xy and x^2y^2 too, taking care to keep $V > 0$ in a neighbourhood of the origin while $V(0, 0) = 0$.

To check for asymptotic stability, assume that (x, y) does not converge to the origin as $t \to \infty$. The trajectory is bounded for positive t because $\dot{V} \leq 0$. Hence, by the Poincaré-Bendixson theorem, the ω–limit set of the trajectory either contains a fixed point, or it is a cycle.

Since $\dot{V} \leq 0$ always, any cycle must satisfy $\dot{V} = 0$. Thus $x = 0$ on the cycle, which is impossible — since $x = 0$ implies $\dot{x} = 2y \neq 0$ unless $(x, y) = (0, 0)$. Clearly, the system has no fixed points apart from the origin. It follows that the origin is asymptotically stable.[4]

Problem 5

Clearly, the origin is the only equilibrium point of the given system.

In the hopes of being able to trap a trajectory in a region bounded away from infinity as well as the origin, we let $V = x^2 + y^2$ and compute

$$\dot{V} = \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = 2x\dot{x} + 2y\dot{y} = -2y^2 - 2y^4 + \frac{4y^2}{1 + x^2}$$

which is certainly positive for small (x, y) and negative for large (x, y).

To be more specific, assume for example $x^2 + y^2 \leq \frac{1}{3}$. Then in particular $x^2 \leq \frac{1}{3}$ and $y^2 \leq \frac{1}{3}$, so $\dot{V} \geq -2y^2 - 2y^4 + 3y^2 = y^2(1 - 2y^2) \geq \frac{1}{3}y^2 \geq 0$.

On the other hand, if $x^2 \geq 1$ then $\dot{V} \leq -2y^4 \leq 0$, so the remaining problem is what happens when $x^2 < 1$ and $|y|$ is large. In any case however $\dot{V} \leq 2y^2 - 2y^4 = 2y^2(1 - y^2) \leq 0$ if $|y| \geq 1$. Now if $x^2 + y^2 \geq 2$ then either $x^2 \geq 1$ or $y^2 \geq 1$, so $\dot{V} \leq 0$ in any case.

We conclude that the annulus $\frac{1}{3} \leq x^2 + y^2 \leq 2$ is forward invariant for the system, and since this annulus contains no equilibrium point, it must contain a cycle by the Poincaré–Bendixson theorem.

Problem 6

The given system corresponds to a vector field $f(x, y) + k(x, y)$ where $f(x, y) = (-y^3, -x^3)$ and $k(x, y) = (g(x, y), h(x, y)$. Now for large (x, y) we find that f dominates, as $|f(x, y)|^2 = x^6 + y^6 \geq mr^6$ where m is the (positive!) minimum[5] of $\cos^6\theta + \sin^6\theta$, while $|k(x, y)|$ is bounded by a second degree polynomial in $r = \sqrt{x^2 + y^2}$. Thus, for r large enough, $|f(x, y)| > |k(x, y)|$ and so the index of a circle of radius r wrt. $f + k$ is equal to that wrt. f.

Now, consider $f(r\cos\theta, r\sin\theta) = -r^3(\sin^3\theta, \cos^3\theta)$. As θ grows from 0 to 2π, $f(r\cos\theta, r\sin\theta)$ starts at the negative y axis, and then travels through the third quadrant, then the second, then the first, and finally through the fourth quadrant back to the negative y axis. Thus the index is -1. Since this is nonzero, there must be at least one equilibrium point in the interior.

[4] We have in fact proved that *every* trajectory converges to the origin as $t \to \infty$. A more elementary proof of asymptotic stability is probably not hard to find, once the Liapunov function has been determined.

[5] $m = 1/4$

If there is only one equilibrium point, it must have index -1. If it is nondegenerate then it must be a saddle point.

Problem 7

a Assume x and y are solutions. Then

$$x(t) = x_0 + \int_0^t f(x(s))\,\mathrm{d}s,$$
$$y(t) = x_0 + \int_0^t f(y(s))\,\mathrm{d}s,$$

from which we conclude

$$\begin{aligned} |x(t) - y(t)| &= \Big| \int_0^t \big(f(x(s)) - f(y(s))\big)\,\mathrm{d}s \Big| \\ &\le \int_0^t |f(x(s)) - f(y(s))|\,\mathrm{d}s \\ &\le L \int_0^t |x(s) - y(s)|\,\mathrm{d}s \end{aligned}$$

where L is the Lipschitz constant of f. Now the Grönwall inequality implies $|x(t) - y(t)| = 0$.

For the counterexample, try for example $\dot{x} = x^{1/3}$ and $x_0 = 0$. One solution is $x(t) = 0$ for all t. Another is $x(t) = 0$ for $t < 0$, but $x(t) = \big(\frac{2}{3}t\big)^{3/2}$ for $t > 0$.

b The flow of f is a function φ so that $\varphi_t(x_0)$ is defined by solving the initial value problem $\dot{x} = f(x)$ with $x(0) = x_0$ and setting $\varphi_t(x_0) = x(t)$. Another way to put this is by the defining equations

$$\frac{\partial}{\partial t}\varphi_t(x) = f\big(\varphi_t(x)\big), \quad \varphi_0(x) = x.$$

Intuitively, start at x and follow the flow for a time t to get to $\varphi_t(x)$.

The basic properties of the flow are, apart from the differential equation it satisfies,

$$\varphi_0(x) = x, \quad \varphi_s(\varphi_t(x)) = \varphi_{s+t}(x).$$

For the vector field $f(x) = x^2$ on $\mathbb{R}$, the general solutions are $x = 1/(t_0 - t)$ in addition to the special solution $x(t) = 0$. This yields

$$\varphi_t(x) = \begin{cases} 0, & x = 0, \\ \dfrac{1}{\dfrac{1}{x} - t}, & x \neq 0. \end{cases}$$

In particular $\varphi_t(x)$ is not defined when $t = 1/x$, nor when t and 0 are on opposite sides of $1/x$. The reason is that the solution tends to infinity in finite time.