Problem 1 Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Sketch the phase diagram of the system with orientations.

b) Find the general solution to the system and identify those initial data $x(0) = x_0$ such that $|x(t)| \to 0$ as $t \to \infty$.

Problem 2 Consider the system

$$\begin{aligned} \dot{x} &= x + y + 2 \\ \dot{y} &= x + y^2. \end{aligned}$$

a) Find and classify all equilibrium points of the system.

b) Sketch the phase diagram of the system with orientations.

Problem 3 Consider the system

$$\begin{aligned} \dot{x} &= x^3 g(x, y) + y^4 x, \\ \dot{y} &= y g(x, y) - x^2 y, \end{aligned}$$

where $g(x, y) = x^2 + y^2 - 4$.
Determine if the origin is a stable, asymptotically stable or unstable equilibrium point of the system.

Problem 4 Consider the system

$$\begin{aligned} \dot{x} &= (x-1)^2 + y^2 - 4 \\ \dot{y} &= (x+1)^2 + y^2 - 4. \end{aligned}$$

Determine whether or not there exists a periodic orbit surrounding all equilibrium points of the system.

Problem 5 Consider the differential equation

$$\dot{x} = ax + \sin(t), \quad x(0) = x_0 \in \mathbb{R}, \quad a \in \mathbb{R}. \tag{1}$$

Show that if $a < 0$ then the solution to (1) remains bounded for all $t \geq 0$.

Problem 6 Consider the initial value problem

$$\dot{x} = (1+t^2)x^2, \quad x(t_0) = x_0 \in \mathbb{R}.$$

Find the general solution.
Determine for which pairs (t_0, x_0) there exists a global solution.

Problem 7 Consider the following system (in polar cocordinates)

$$\dot{r} = r(r-1) \tag{2a}$$
$$\dot{\theta} = 2 \tag{2b}$$

and let Σ be the Poincaré section

$$\Sigma = \{(0, y) \in \mathbb{R}^2 \mid y > 0\}.$$

Determine the Poincaré map $\Pi(\Sigma) : \Sigma \to \Sigma$ and use it to show that the unit circle represents an unstable periodic solution.

Hint: The solution to (2a) is given by $r(t) = \frac{r(0)}{e^t + r(0)(1-e^t)}$.