1a) LET $A=\begin{pmatrix} 0 & -2 \\ 4 & 6 \end{pmatrix}$

EIGENVALUES $\det(A-\lambda I)=0$: $-\lambda(6-\lambda)+8=0$

$$\lambda^2-6\lambda+8=0$$

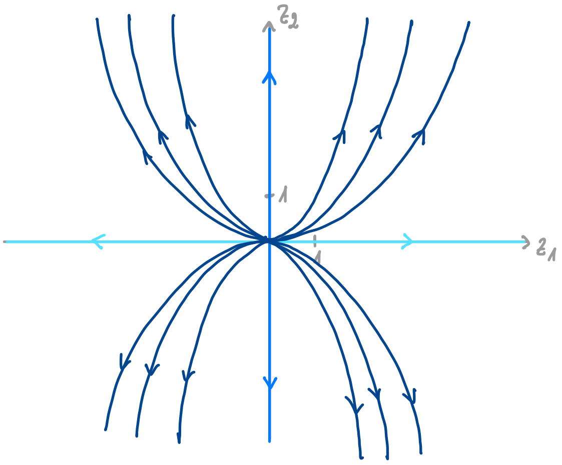
$$\lambda_{1,2}=3\pm\sqrt{9-8}=3\pm 1 \implies \lambda_1=2,\ \lambda_2=4$$

EIGENVECTORS: $\lambda_1=2$ $(A-2I)\vec{v}_1=\begin{pmatrix} -2 & -2 \\ 4 & 4 \end{pmatrix}\vec{v}_1=\vec{0} \Rightarrow \vec{v}_1=c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $(c\in\mathbb{R})$

$\lambda_2=4$: $(A-4I)\vec{v}_2=\begin{pmatrix} -4 & -2 \\ 4 & 2 \end{pmatrix}\vec{v}_2=\vec{0} \Rightarrow \vec{v}_2=d\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $(d\in\mathbb{R})$

LET $P=[\vec{v}_1,\vec{v}_2]=\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$, $D=\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow A=PDP^{-1}$

- PHASE DIAGRAM FOR $\dot{\vec{z}}=D\vec{z}$ WHERE $\vec{z}=\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

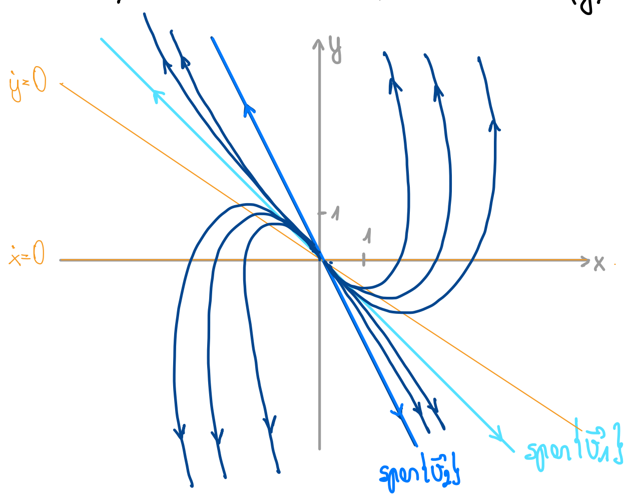


$z_1(t)=e^{2t}z_1(0)$
$z_2(t)=e^{4t}z_2(0)$

$$\Rightarrow z_2(t)=e^{4t}z_2(0)=\left(e^{2t}z_1(0)\right)^2\frac{z_2(0)}{z_1^2(0)}=z_1^2(t)\frac{z_2(0)}{z_1^2(0)}$$

$\rightsquigarrow$ OF THE FORM: $z_2=az_1^2$ $(a\in\mathbb{R},\ z_1\neq 0)$

- PHASE DIAGRAM FOR $\dot{\vec{x}}=A\vec{x}$ WHERE $\vec{x}=\begin{pmatrix} x \\ y \end{pmatrix}$



ISOCLINES:

- $\dot{x}=0$: $-2y=0 \Rightarrow y=0$
- $\dot{y}=0$: $4x+6y=0 \Rightarrow y=-\frac{2}{3}x$

1b) GENERAL SOLUTION

$$\vec{x}(t)=c_1e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}+c_2e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad (c_1,c_2\in\mathbb{R})$$

SINCE $\lambda_1=2$ & $\lambda_2=4$ ARE POSITIVE: $\vec{x}(t)\to\vec{0} \iff \vec{x}_0=\vec{0}$

2a) EP: $\dot{x}=0 \Rightarrow y=-x-2$, $\dot{y}=0 \Rightarrow y^2=-x$ $\Bigg\} \Rightarrow$ $y^2-y-2=0$, $y^2=-x$ $\Bigg\} \Rightarrow$ $y_{1,2}=\frac{1}{2}\pm\sqrt{\frac{1}{4}+2}=\frac{1}{2}\pm\frac{3}{2}$, $y^2=-x$

$\Rightarrow$ TWO EP: $(-1,-1)$, $(-4,2)$

JACOBIAN: $Df(\vec{x})=\begin{pmatrix} 1 & 1 \\ 1 & 2y \end{pmatrix}$ [$f(\vec{x})$... RHS OF THE SYSTEM, IE $\dot{\vec{x}}=f(\vec{x})$]

•) $Df((-1,-1))=\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$: EIGENVALUES:

$$\det(Df((-1,-1))-\lambda I)=0$$
$$(1-\lambda)(-2-\lambda)-1=0$$
$$-2+2\lambda-\lambda+\lambda^2-1=0$$
$$\lambda^2+\lambda-3=0$$
$$\lambda_{1,2}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+3} \Rightarrow \lambda_1<0 \ \& \ \lambda_2>0$$

$\Rightarrow (-1,-1)$... SADDLE POINT

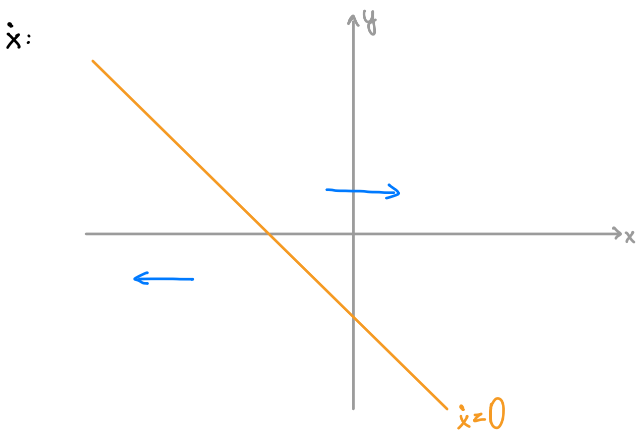
•) $Df((-4,2))=\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$: EIGENVALUES:

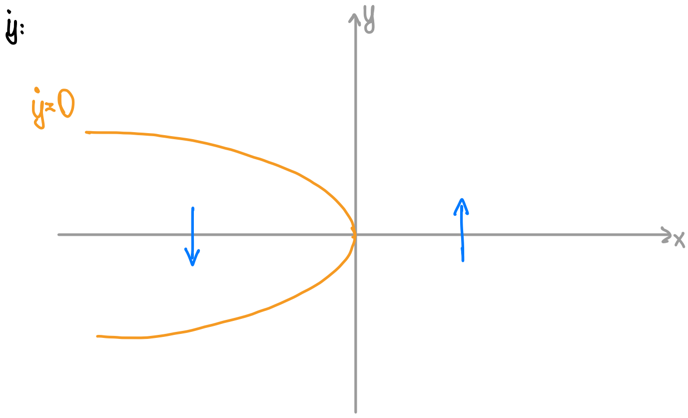
$$\det(Df((-4,2))-\lambda I)=0$$
$$(1-\lambda)(4-\lambda)-1=0$$
$$4-4\lambda-\lambda+\lambda^2-1=0$$
$$\lambda^2-5\lambda+3=0$$
$$\lambda_{1,2}=\frac{5}{2}\pm\sqrt{\frac{25}{4}-\frac{12}{4}} \Rightarrow \lambda_1,\lambda_2>0$$

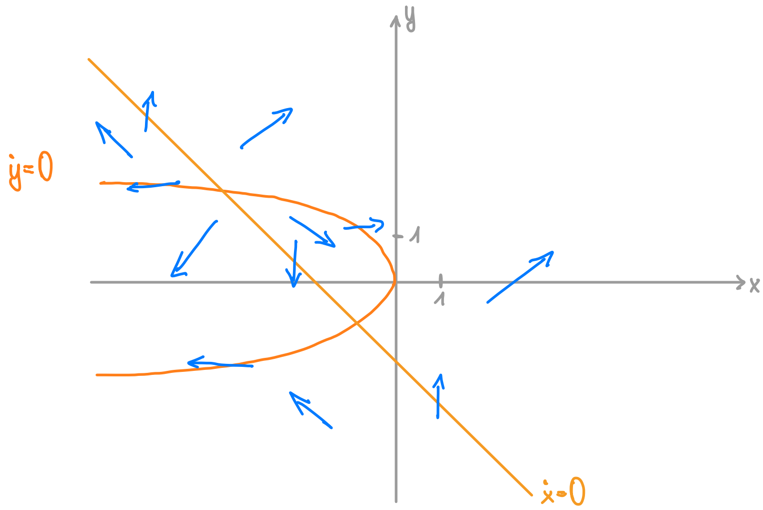
$\Rightarrow (-4,2)$... UNSTABLE NODE

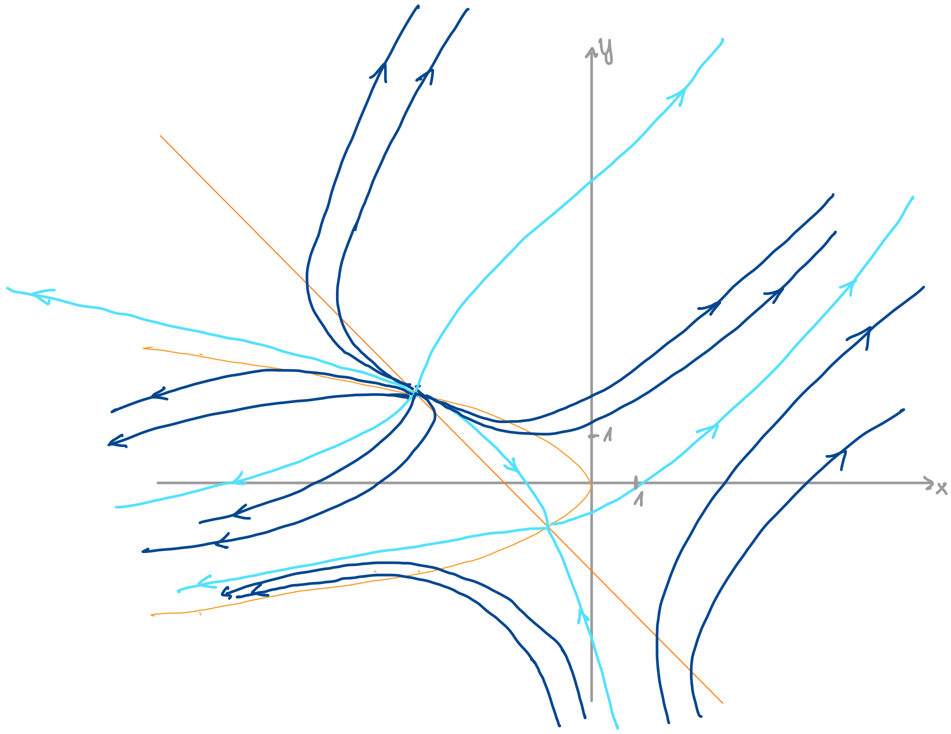
2b) FROM a) EP: $(-1,-1)$... SADDLE
$(-4,2)$... UNSTABLE NODE

•) ISOCLINES: $\dot{x}=0 \Rightarrow y=-x-2$
$\dot{y}=0 \Rightarrow x=-y^2$









3) ·) LINEARISATION AT (0,0)

$$\vec{x}' = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \vec{x} = A\vec{x}$$

EIGENVALUES OF A: 0, -4

⤳ CAN'T USE LINEARISATION

·) LYAPUNOV FCT

LET $V(x,y) = 2x^2 + y^4$ ... CONTINUOUS, STRICT MIN AT (0,0), $V(0,0) = 0$

$$\dot V(x,y) = 4x\dot x + 4y^3 \dot y = 4x(x^3 g(x,y) + y^4 x) + 4y^3(y g(x,y) - x^2 y)$$

$$= 4(x^4 + y^4) g(x,y) < 0$$

$\forall \vec{x} = (x,y)$ ST $0 < |\vec{x}| \leq 1$

[LET $\vec{x} = (x,y)$ ST $|\vec{x}| \leq 1 \Rightarrow g(x,y) = x^2 + y^2 - 4 \leq -3 < 0$]

$\dot{V}(0,0)=0$

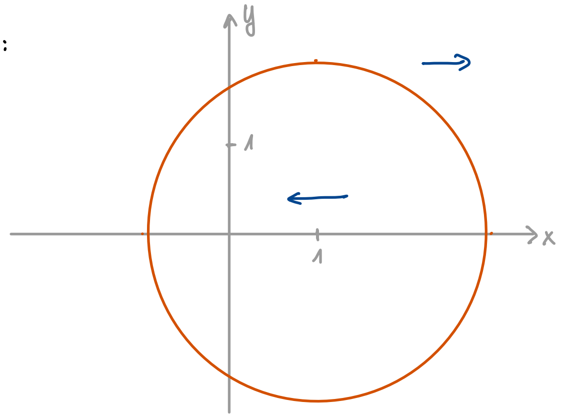
$\Rightarrow (0,0)$ STABLE AND ASYMPTOTICALLY STABLE.

4) EP: $\dot{x}=0 \Rightarrow y^2-4=-(x-1)^2$, $\dot{y}=0 \Rightarrow y^2-4=-(x+1)^2$ $\Rightarrow$ $y^2-4=-(x-1)^2$, $x-1=\pm(x+1)$ $\Rightarrow$ $y^2-4=-(x-1)^2$, $x=0$ $\Rightarrow$ $y^2-4=-1$, $x=0$

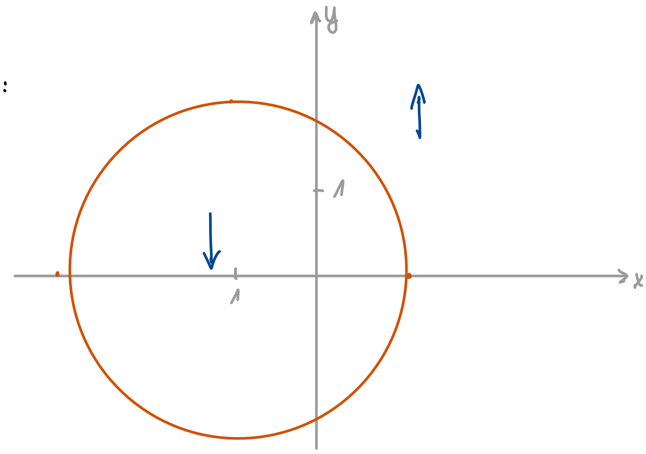
$\Rightarrow$ TWO EPs: $(0,\pm\sqrt{3})$

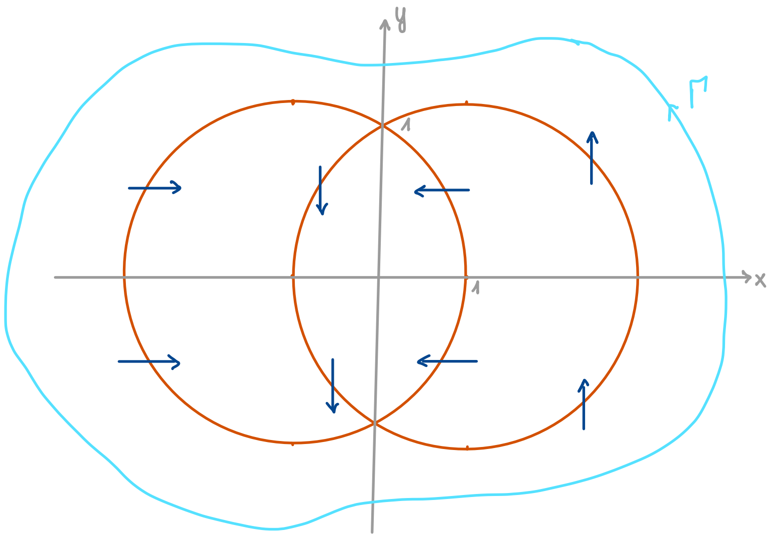
ISOCLINES / NULLCLINES:

$\dot{x}$:



$\dot{y}$:





LET $\vec{f}(\vec{x})=\begin{pmatrix}(x-1)^2+y^2-4\\(x+1)^2+y^2-4\end{pmatrix}$

INDEX:

$I_f(\Gamma)=0$

$\Rightarrow$ THE INDEX OF EVERY CURVE SURROUNDING ALL EP EQUALS ZERO.

INDEX OF A PERIODIC ORBIT SURROUNDING ALL EP: 1

$\Rightarrow$ THERE EXISTS NO PERIODIC ORBIT SURROUNDING ALL EPs.

5) $\dot{x}=ax+\sin(t)$

LET $y(t)=e^{-at}x(t) \Rightarrow \dot{y}(t)=(\dot{x}(t)-ax(t))e^{-at}=e^{-at}\sin(t)$

$$\Rightarrow y(t) = y(0) + \int_0^t e^{-as} \sin(s)\,ds =$$

$$\Rightarrow x(t) = e^{at} y(t) = e^{at} y(0) + \int_0^t e^{a(t-s)} \sin(s)\,ds = e^{at} x(0) + \int_0^t e^{a(t-s)} \sin(s)\,ds.$$

- $|e^{at} x(0)| \leq |x(0)|$ SINCE $a<0$ & $t \geq 0$
- $\left| \int_0^t e^{a(t-s)} \sin(s)\,ds \right| \leq \int_0^t e^{a(t-s)}\,ds = -\frac{1}{a} e^{a(t-s)} \Big|_0^t = \frac{1}{a}\left(e^{at} - 1\right)$

$$\Rightarrow \left| \int_0^t e^{a(t-s)}\,ds \right| \leq \frac{1}{|a|}\left(1 - e^{at}\right) \leq \frac{1}{|a|} \quad \text{SINCE } a<0 \text{ \& } t \geq 0$$

$$\Rightarrow |x(t)| \leq |e^{at} x(0)| + \left| \int_0^t e^{a(t-s)} \sin(s)\,ds \right| \leq |x(0)| + \frac{1}{|a|}$$

$\Rightarrow x(t)$ REMAINS BOUNDED FOR ALL $t \geq 0$.

6) $\dot{x} = (1+t^2)x^2$

$x_0 = 0 \Rightarrow x(t) = 0 \quad \forall t$

$x_0 \neq 0$: $\quad \frac{\dot{x}}{x^2} = 1+t^2 \Rightarrow -\frac{1}{x(t)} + \frac{1}{x(t_0)} = \int_{t_0}^t 1+s^2 = t + \frac{t^3}{3} - t_0 - \frac{t_0^3}{3}$

$$\Rightarrow x(t) - x(t_0) = x(t_0)x(t)\left(t + \frac{t^3}{3} - t_0 - \frac{t_0^3}{3}\right)$$

$$\Rightarrow x(t) = \frac{x(t_0)}{1 - x(t_0)\left(t + \frac{t^3}{3} - t_0 - \frac{t_0^3}{3}\right)} = \frac{x_0}{1 - x_0\left(t + \frac{t^3}{3} - t_0 - \frac{t_0^3}{3}\right)}$$

$f(t) = t + \frac{t^3}{3} \Rightarrow f'(t) = 1 + t^2$ ... $f(t)$ STRICTLY INCREASING

$f(t) \to \pm\infty \quad (t \to \pm\infty)$

$\Rightarrow$ IF $x_0 \neq 0$, THEN $\exists t^* \in \mathbb{R}$ ST $1 - x_0\left(t + \frac{t^3}{3} - t_0 - \frac{t_0^3}{3}\right) = 0$

$\Rightarrow |x(t)| \to \infty$ AS $t \to t^*$

$\Rightarrow$ NO GLOBAL SOLUTION

$\Rightarrow$ ONLY FOR $(x_0, t_0) = (0, t_0)$ WITH $t_0 \in \mathbb{R}$ THERE EXISTS A GLOBAL SOLUTION.

f) $\Sigma = \{(0,y) \in \mathbb{R}^2 \mid y > 0\}$, $\vec{x} \in \Sigma \Rightarrow \vec{x} = (r\cos(\theta), r\sin(\theta)) = (0,r)$ WHERE $r = |\vec{x}|$ AND $\theta = 2n\pi + \frac{\pi}{2}$ $(n \in \mathbb{Z})$

$\Rightarrow$ IF $\vec{x}(0) = \vec{x}_0 \in \Sigma$, THEN $\vec{x}(t, \vec{x}_0)$ WILL LEAVE Σ BUT RETURN TO Σ IF θ HAS CHANGED BY 2π, SINCE $\dot{\theta} = 2 > 0$.

$\Rightarrow$ THE TIME $\tau(\vec{x}_0)$ AT WHICH $\vec{x}(t, \vec{x}_0)$ RETURNS TO Σ FOR THE FIRST TIME IS GIVEN BY $\theta(\tau(\vec{x}_0)) - \theta(0) = 2\tau(\vec{x}_0) = 2\pi$

$\Rightarrow \tau(\vec{x}_0) = \pi$

$\Rightarrow \Pi(\vec{x}_0) = \vec{x}(\tau(\vec{x}_0); \vec{x}_0) = \vec{x}(\pi, \vec{x}_0)$ ... POINCARÉ MAP

$\Rightarrow$ $(\vec{x}_0 = (0, r_0))$: $\Pi((0,r_0)) = \Pi(\vec{x}_0) = \vec{x}(\pi, \vec{x}_0) = (0, r(\pi))$ WHERE $r(t)$ IS THE SOL. TO (2a)

$$\Rightarrow \Pi((0,r_0)) = (0, r(\pi)) = \left(0, \frac{r_0}{e^{\pi} + r_0(1-e^{\pi})}\right) = \left(0, \frac{r_0}{r_0 + e^{\pi}(1-r_0)}\right)$$

NOTE $r(\pi) < r_0$ IF $0 < r_0 < 1$ AND $r(\pi) > r_0$ IF $1 < r_0 < -\frac{e^{\pi}}{1-e^{\pi}}$

$\ulcorner r_0 + e^{\pi}(1-r_0) = 0 \Rightarrow r_0(1-e^{\pi}) + e^{\pi} = 0 \Rightarrow r_0 = -\frac{e^{\pi}}{1-e^{\pi}} \lrcorner$

$\Rightarrow |r(\pi) - 1| > |r_0 - 1|$ IF $r_0 \neq 1$

$\Rightarrow (0,1) \notin \omega(\Gamma_{\vec{x}_0})$ $\forall \vec{x}_0 \in \Sigma$ ($\Gamma_{\vec{x}_0}$... ORBIT THROUGH $\vec{x}_0$)

$\Rightarrow \Gamma_{(0,1)}$ IS UNSTABLE.