Problem 1 Consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Sketch the phase diagram of the system with orientations.

b) Find the general solution to the system and the one satisfying $\mathbf{x}(1) = \begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-2} \end{pmatrix}$.

Problem 2 Consider the system

$$\begin{aligned} x' &= y - 2 \\ y' &= x^2 + y - 3. \end{aligned}$$

a) Find and classify all equilibrium points of the system.

b) Sketch the phase diagram of the system with orientations.

Problem 3 Consider the system

$$\begin{aligned} x' &= -x^3 - y \\ y' &= x^3 - y^3. \end{aligned}$$

Find a value $\beta \in \mathbb{R}$ such that

$$L(x, y) = x^4 + 2\beta y^2, \quad x, y \in \mathbb{R},$$

is a strict Lyapunov function for the origin. What can you conclude then?

Problem 4 Consider the system

$$\begin{aligned} x' &= \frac{x^3}{1 + x^2 + y^2} \\ y' &= \frac{y^3}{1 + x^2 + y^2}. \end{aligned}$$

Determine whether or not this system has non-constant periodic solutions.

Problem 5 Consider the system

$$x' = (y - x)(y + x)$$
$$y' = x(x + 1).$$

a) Compute the index of the curve $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$.

b) Determine whether or not the system has closed phase paths surrounding all equilibrium points.

Problem 6 Given the initial value problem

$$x' = x^2 + t^2$$
$$x(0) = x_0 > 0.$$

Use a comparison argument to show that there exists $t^* > 0$ such that $\lim_{t \uparrow t^*} x(t) = \infty$.

Problem 7 Show that if $g : [0, T] \to \mathbb{R}$ is continuous and if there exists a positive constant C such that

$$g(t) \leq C + \int_0^t s g(s) ds \quad \text{for all } 0 \leq t \leq T$$

then

$$g(t) \leq C e^{\frac{t^2}{2}} \quad \text{for all } \ 0 \leq t \leq T.$$