1a) $\vec{x}' = A\vec{x}$ WHERE $A = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix}$

EIGENVALUES OF A:

$$\det(A-\lambda I) = (-3-\lambda)^2 - 1 = (3+\lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda + 3 = \pm 1$$

$$\Rightarrow \lambda_1 = -2 \text{ AND } \lambda_2 = -4 \quad \rightsquigarrow (0,0) \ldots \text{STABLE NODE}$$

EIGENVECTORS: $\lambda_1 = -2$:

$$(A + 2I)\vec{v}_1 = 0$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \vec{v}_1 = 0 \quad \Rightarrow \vec{v}_1 = d\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (d \in \mathbb{R})$$

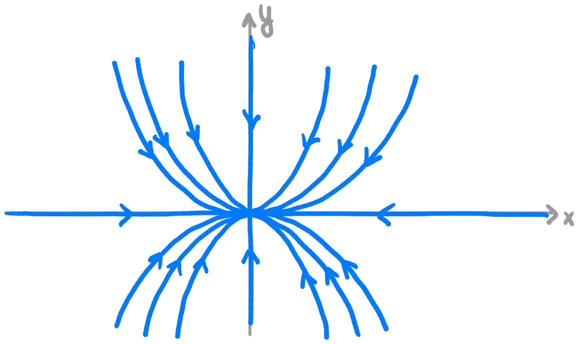
$\lambda_1 = -4$

$$(A + 4I)\vec{v}_2 = 0$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \vec{v}_2 = 0 \quad \Rightarrow \vec{v}_2 = d\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (d \in \mathbb{R})$$

$$\Rightarrow A = SDS^{-1} \quad \text{WHERE } D = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \text{ AND } S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

PHASE DIAGRAM FOR $\vec{x}' = D\vec{x}$



$$x(t) = x(0)e^{-2t}$$

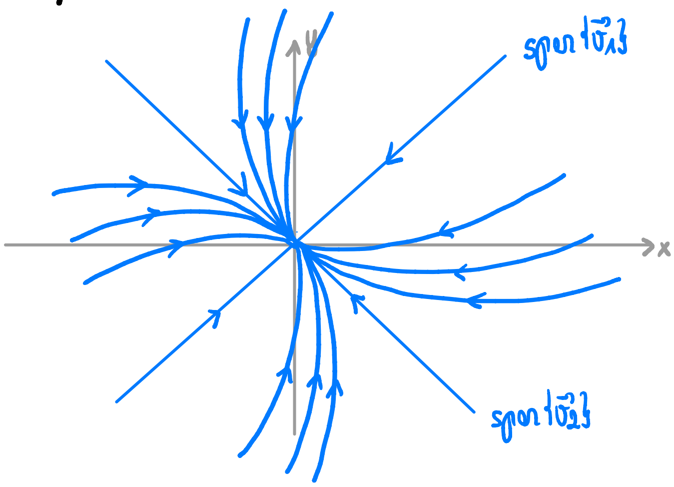
$$y(t) = y(0)e^{-4t}$$

$$\Rightarrow y(t) = \frac{y(0)}{x(0)^2} x(t)^2 \quad \text{IF } x(0) \neq 0$$

$$\Rightarrow y(x) = ax^2 \quad \text{IF } x(0) \neq 0$$

$$\text{IF } \vec{x}(0) = 0: \Rightarrow y\text{-AXIS}$$

PHASE DIAGRAM FOR $\vec{x} = A\vec{x}$:



1b) GENERAL SOLUTION

$$\vec{x}(t) = \underbrace{C_1 e^{-2t}}_{e^{\lambda_1 t}} \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{v_1}} + \underbrace{C_2 e^{-4t}}_{e^{\lambda_2 t}} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{v_2}} \quad \text{WHERE } C_1, C_2 \in \mathbb{R}$$

$$\vec{x}(1) = \begin{pmatrix} 0 \\ e^{-2} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ e^{-2} \end{pmatrix} = C_1 e^{-2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{-4} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} C_1 e^{-2} + C_2 e^{-4} &= 0 \\ -C_1 e^{-2} + C_2 e^{-4} &= e^{-2} \end{aligned}$$

$$\Rightarrow \begin{aligned} C_1 e^{-2} + C_2 e^{-4} &= 0 \\ 2C_2 e^{-4} &= e^{-2} \end{aligned}$$

$$\Rightarrow C_2 = \frac{1}{2} e^{2} \quad \text{AND} \quad C_1 = -C_2 e^{-2} = -\frac{1}{2}$$

$$\Rightarrow \vec{x}(t) = -\frac{1}{2} e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2} e^{2-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{SATISFIES } \vec{x}(1) = \begin{pmatrix} 0 \\ e^{-2} \end{pmatrix}$$

2a)
$$\begin{aligned} x' &= y - 2 \\ y' &= x^2 + y - 3 \end{aligned}$$

EP: $\begin{aligned} x' &= 0 \\ y' &= 0 \end{aligned} \Rightarrow \begin{aligned} y - 2 &= 0 \\ x^2 + y - 3 &= 0 \end{aligned} \Rightarrow \begin{aligned} y &= 2 \\ x^2 &= 3 - y = 1 \end{aligned} \Rightarrow$ EP: $(\pm 1, 2)$

LET $F(x,y) = \begin{pmatrix} y - 2 \\ x^2 + y - 3 \end{pmatrix}$

$$\Rightarrow DF((x,y)) = \begin{pmatrix} 0 & 1 \\ 2x & 1 \end{pmatrix}$$

- EP $(-1,2)$: $DF((-1,2)) = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} \rightsquigarrow$ EIGENVALUES:

$$\det(DF((-1,2)) - \lambda I) = -\lambda(1-\lambda) + 2 = 0$$

$$\lambda^2 - \lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\underbrace{\frac{1}{4} - 2}_{<0}}$$

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2} i$$

$\Rightarrow (-1,2)$ UNSTABLE SPIRAL

•) EP $(1,2)$:

$$DF((1,2)) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \rightsquigarrow$$

EIGENVALUES:

$$\det(DF((1,2)) - \lambda I) = -\lambda(1-\lambda) - 2 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

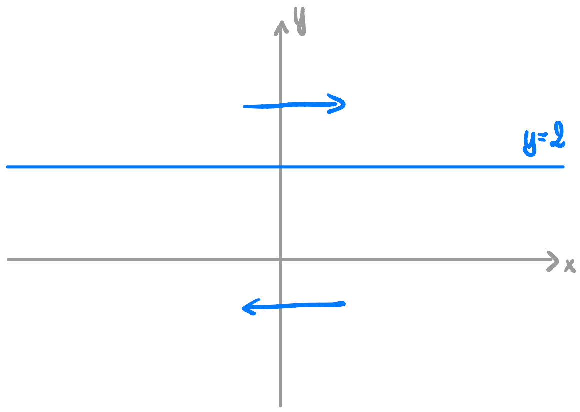
$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}$$

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{3}{2}$$

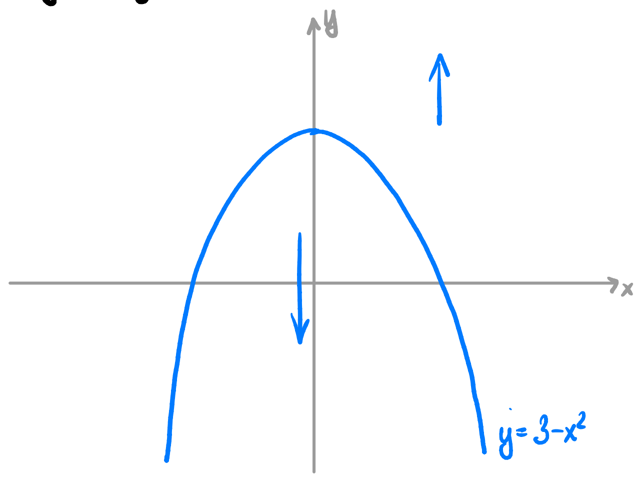
$\lambda_1 = -1$ AND $\lambda_2 = 2$

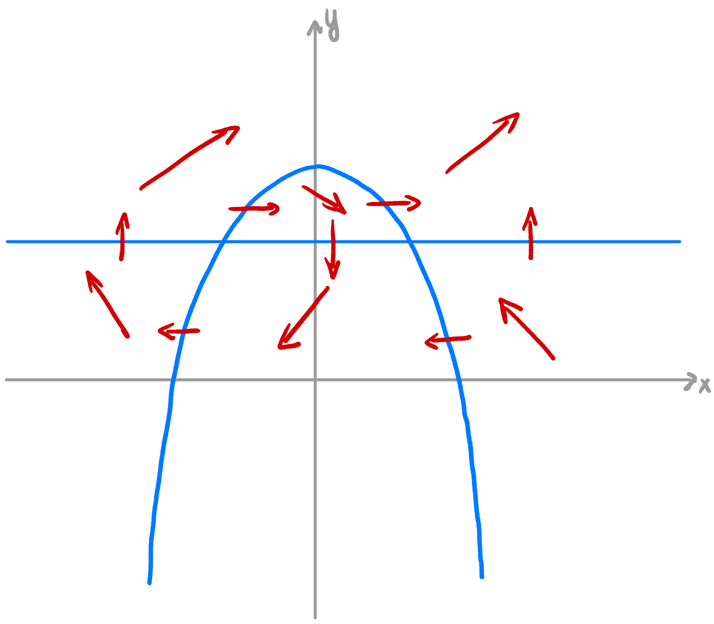
$\Rightarrow (1,2)$... SADDLE

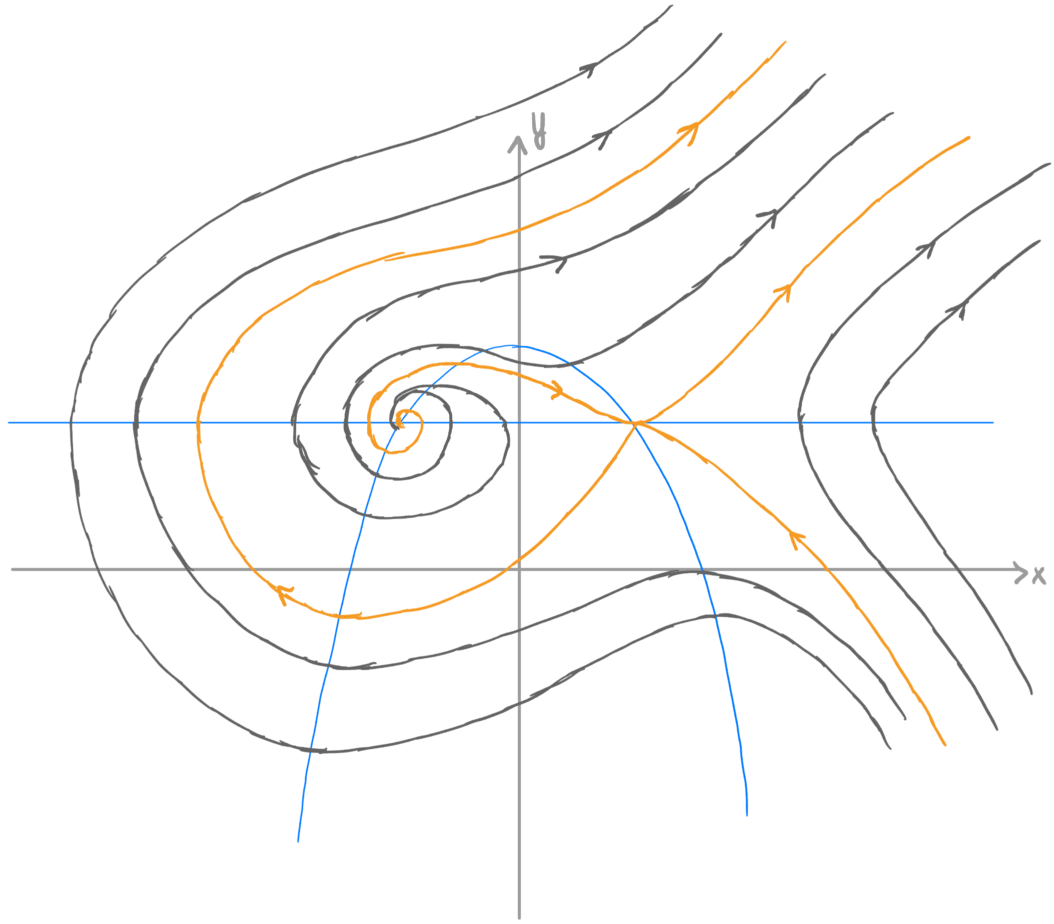
2b) $x' = y - 2$



$y' = x^2 + y - 3$







3) $L(x,y) = x^4 + 2\beta y^2$ IS A STRICT LYAPUNOV FCT FOR $(0,0) = \vec{0}$ IF

- $L(x,y) > L(0,0) = 0 \quad \forall\, (x,y) \neq \vec{0}$
- $\frac{d}{dt} L(x(t), y(t)) < 0 \quad \forall\, (\tilde{x}(t), \tilde{y}(t)) \neq \vec{0}$ SOL TO THE GIVEN SYSTEM

•) $L(0,1) = 2\beta > 0 \Rightarrow \beta > 0$

$\Rightarrow L(x,y) > 0 \quad \forall\, (x,y) \neq \vec{0}$

•) $\frac{d}{dt} L(x(t), y(t)) = 4x^3(t)\dot{x}(t) + 4\beta y(t)\dot{y}(t)$

$= -4x^6(t) - 4x^3(t)\, y(t) + 4\beta x^3(t) y(t) - 4\beta y^4(t)$

$= -4\left(x^6(t) + (1-\beta)x^3(t)\, y(t) + \beta y^4(t)\right)$

IF $\beta = 1 \Rightarrow \frac{d}{dt} L(x(t), y(t)) = -4\left(x^6(t) + y^4(t)\right) < 0 \quad \forall\, (x(t), y(t)) \neq \vec{0}$

$\Rightarrow L(x,y) = x^4 + 2y^2$ IS A STRICT LYAPUNOV FCT FOR $(0,0)$.

$\Rightarrow (0,0)$ IS AN ASYMPTOTICALLY STABLE EP.

4) CHOOSE $\varphi(x,y) = 1 + x^2 + y^2$

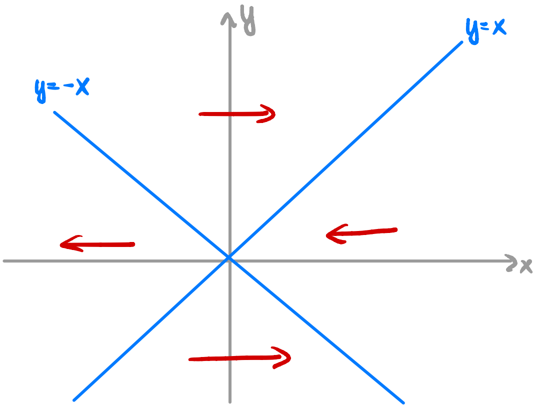
AND LET
$$F(x,y) = \begin{pmatrix} \frac{x^3}{1+x^2+y^2} \\ \frac{y^3}{1+x^2+y^2} \end{pmatrix}$$

$$\Rightarrow \varphi F(x,y) = \begin{pmatrix} x^3 \\ y^3 \end{pmatrix} \Rightarrow \nabla \cdot (\varphi F)(x,y) = 3(x^2+y^2) > 0 \quad \forall (x,y) \neq \vec{0}$$
$$\nabla \cdot (\varphi F)(0,0) = 0$$

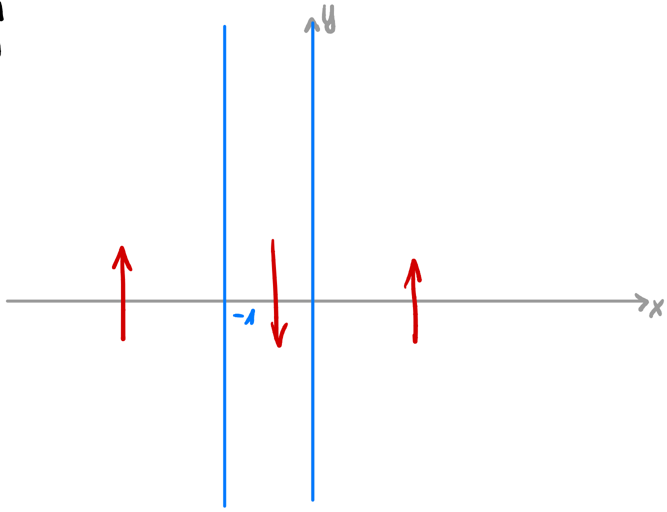
$\Rightarrow$ (DULAC): THE SYSTEM HAS NO NON-CONSTANT PERIODIC SOL.

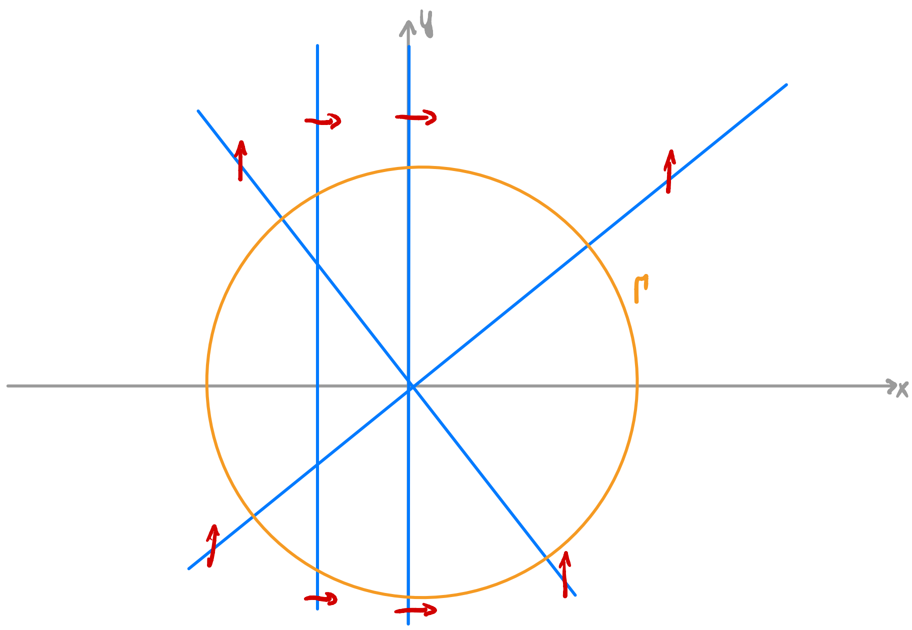
5) $x' = (y-x)(y+x)$
$y' = x(x+1)$

x':



y'





$\Rightarrow \uparrow \rightsquigarrow \rightarrow \rightsquigarrow \rightarrow \rightsquigarrow \uparrow \rightsquigarrow \uparrow \rightsquigarrow \rightarrow \rightsquigarrow \rightarrow \rightsquigarrow \uparrow \rightsquigarrow \uparrow \Rightarrow$ 0 TURNS COUNTERCLOCKWISE
$\Rightarrow$ INDEX: 0

b.) EP: $\begin{array}{l} x' = 0 \\ y' = 0 \end{array} \Rightarrow \begin{array}{l} x = y \text{ OR } x = -y \\ x = 0 \text{ OR } x = -1 \end{array} \Rightarrow$ EP: $(0,0), (-1, \pm 1)$

$\Rightarrow$ ALL EP LIE INSIDE Γ

$\Rightarrow$ EVERY SIMPLE, CLOSED CURVE SURROUNDING THESE 3 EP HAS THE SAME INDEX AS Γ, IE 0

EVERY CLOSED PHASE PATH SURROUNDING ALL EP IS A SIMPLE CLOSED CURVE WITH INDEX 1 SURROUNDING ALL EP. $\lightning$.

$\Rightarrow$ NO CLOSED PHASE PATH SURROUNDING ALL EP.

6) $x(t)$ SOLUTION TO $\begin{array}{l} x' = x^2 + t^2 \\ x(0) = x_0 > 0. \end{array}$ $(*)$ $\rightsquigarrow$ $x(t) > 0$ $\forall t > 0$ FOR WHICH A SOL EXISTS

$y(t)$ SOLUTION TO $\begin{array}{l} y' = y^2 \\ y(0) = x_0 > 0 \end{array}$ $(**)$

$$\frac{y'(t)}{y^2(t)} = 1 \rightsquigarrow -\frac{1}{y(t)} + \frac{1}{y(0)} = t \Rightarrow y(t) - y(0) = y(0)\, y(t)\, t$$

$$\Rightarrow y(t)(1 - y(0)t) = y(0)$$

$$\Rightarrow y(t) = \frac{y(0)}{1 - y(0)t} \longrightarrow \infty \text{ AS } t \to \frac{1}{y(0)}$$

$\Rightarrow$ $(**)$ HAS ONLY A LOCAL SOLUTION FORWARD IN TIME AND $y(t) > 0$ $\forall\, 0 < t < \frac{1}{y(0)}$.

$$t = \int_0^t \frac{y'(s)}{y^2(s)}\, ds = \int_0^t \frac{x'(s)}{x^2(s) + s^2}\, ds \leq \int_0^t \frac{x'(s)}{x^2(s)}\, ds$$

SINCE $x'(s) \geq 0$

$$\Rightarrow \int_{y(0)}^{y(t)} \frac{1}{z^2}\, dz = \int_0^t \frac{y'(s)}{y^2(s)}\, ds \leq \int_0^t \frac{x'(s)}{x^2(s)}\, ds = \int_{x(0)}^{x(t)} \frac{1}{z^2}\, dz$$

$\Rightarrow$ $(x(0) = y(0) = x_0 > 0)$ $$\int_{x_0}^{y(t)} \frac{1}{z^2}\, dz \leq \int_{x_0}^{x(t)} \frac{1}{z^2}\, dz$$

$\Rightarrow$ $y(t) \leq x(t)$ $\Rightarrow$ $x(t) \to \infty$ AS $t \to t^*$ FOR SOME $t^* < \frac{1}{x_0}$.

4) $$g(t)\le C+\int_0^t s\,g(s)\,ds$$

LET $f(t)=t\,g(t)$

$$\Rightarrow f(t)=t\,g(t)\le Ct+t\int_0^t s\,g(s)\,ds = Ct+t\int_0^t f(s)\,ds$$

$$\Rightarrow f(t)-t\int_0^t f(s)\,ds\le Ct$$

$$\Rightarrow \underbrace{e^{-\frac{t^2}{2}}}_{u}\,\underbrace{f(t)}_{v'}\underbrace{-\,e^{-\frac{t^2}{2}}t}_{u'}\,\underbrace{\int_0^t f(s)\,ds}_{v}\le e^{-\frac{t^2}{2}}Ct$$

$$\left(e^{-\frac{t^2}{2}}\int_0^t f(s)\,ds\right)_t\le C\,t\,e^{-\frac{t^2}{2}}\qquad \Big|\int_0^t$$

$$e^{-\frac{t^2}{2}}\int_0^t f(s)\,ds\le -C\left(e^{-\frac{t^2}{2}}-1\right)\qquad \Big|\,e^{\frac{t^2}{2}}$$

$$\int_0^t f(s)\,ds\le C\left(e^{\frac{t^2}{2}}-1\right)$$

$$\Rightarrow g(t)\le C+\int_0^t s\,g(s)\,ds = C+\int_0^t f(s)\,ds\le Ce^{\frac{t^2}{2}}.$$