Problem 1 Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Sketch the phase diagram of the system with orientations.

b) Find the general solution to the system and the one satisfying $\mathbf{x}(1) = \begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} 2e^2 \\ 0 \end{pmatrix}$.

Problem 2 Consider the system

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= y - x^2 + 2. \end{aligned}$$

a) Find and classify all equilibrium points of the system.

b) Sketch the phase diagram of the system with orientations.

Problem 3 Consider the system

$$\begin{aligned} \dot{x} &= x^3 g(x,y) - y, \\ \dot{y} &= y^3 g(x,y) + x, \end{aligned}$$

where $g(x,y) = 16 - x^2 - (y-2)^2$.

a) Determine if the origin is a stable, asymptotically stable or unstable equilibrium point of the system.

b) Show that there exists a periodic orbit in the annular region $\{\mathbf{x} \in \mathbb{R}^2 \mid 1 \leq |\mathbf{x}| \leq 7\}$.

Problem 4 Consider the system

$$\begin{aligned} \dot{x} &= y(x^2 + y) \\ \dot{y} &= x + y^2. \end{aligned}$$

Compute the index of the origin.

Problem 5 Consider the differential equation

$$\dot{x} = \cos\left(\frac{\pi}{1+\varepsilon^2}x\right), \quad x(0) = \frac{1}{4}, \quad \varepsilon \in \mathbb{R}. \tag{1}$$

a) Given $\varepsilon \in \mathbb{R}$, show that the function $f^\varepsilon(x) = \cos\left(\frac{\pi}{1+\varepsilon^2}x\right)$ is globally Lipschitz continuous by deriving a Lipschitz constant.

For $\varepsilon = 0$, show that there exist constants C_1 and C_2 such that

$$C_1 \leq x(t) \leq C_2 \quad \text{for all } t \in \mathbb{R}.$$

b) Denote by x^0 and x^ε the solution to (1) with $\varepsilon = 0$ and $\varepsilon \neq 0$, respectively. Show, with the help of a Grönwall estimate, that there exists a function $K(t)$ independent of ε such that

$$|x^0(t) - x^\varepsilon(t)| \leq K(t)\varepsilon^2 \quad \text{for all } t \geq 0.$$

Problem 6 Consider the following system (in polar cocordinates)

$$\dot{r} = r(1-r) \tag{2a}$$

$$\dot{\theta} = 2 \tag{2b}$$

and let Σ be the Poincaré section

$$\Sigma = \{(x, 0) \in \mathbb{R}^2 \mid x > 0\}.$$

Determine the Poincaré map $\Pi(\Sigma) : \Sigma \to \Sigma$ and use it to show that the unit circle represents an asymptotically stable periodic solution.

Hint: The solution to (2a) is given by $r(t) = \frac{r(0)}{e^{-t}+r(0)(1-e^{-t})}$.