1a) LET $A=\begin{pmatrix}4 & -6\\ 2 & -4\end{pmatrix}$.

EIGENVALUES: $\det(A-\lambda I)=0$: $(4-\lambda)(-4-\lambda)+12=0$

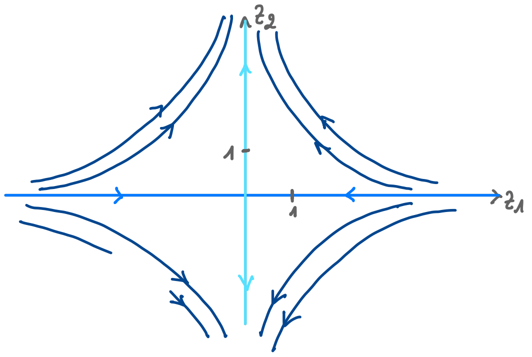
$$\lambda^2-16+12=0$$
$$\lambda^2=4$$
$$\lambda_{1,2}=\pm 2$$

EIGENVECTORS: $\lambda_1=-2$ $(A+2I)\vec{v}_1=\begin{pmatrix}6 & -6\\ 2 & -2\end{pmatrix}\vec{v}_1=\vec{0} \Rightarrow \vec{v}_1=c\begin{pmatrix}1\\1\end{pmatrix}$ $(c\in\mathbb{R})$

$\lambda_2=2$: $(A-2I)\vec{v}_2=\begin{pmatrix}2 & -6\\ 2 & -6\end{pmatrix}\vec{v}_2=\vec{0} \Rightarrow \vec{v}_2=d\begin{pmatrix}3\\1\end{pmatrix}$ $(d\in\mathbb{R})$

$\Rightarrow$ LET $P=[\vec{v}_1,\vec{v}_2]=\begin{pmatrix}1 & 3\\ 1 & 1\end{pmatrix}$, $D=\begin{pmatrix}-2 & 0\\ 0 & 2\end{pmatrix}$ $\Rightarrow$ $A=PDP^{-1}$

·) PHASE DIAGRAM FOR $\dot{\vec{z}}=D\vec{z}$ WHERE $\vec{z}=\begin{pmatrix}z_1\\z_2\end{pmatrix}$

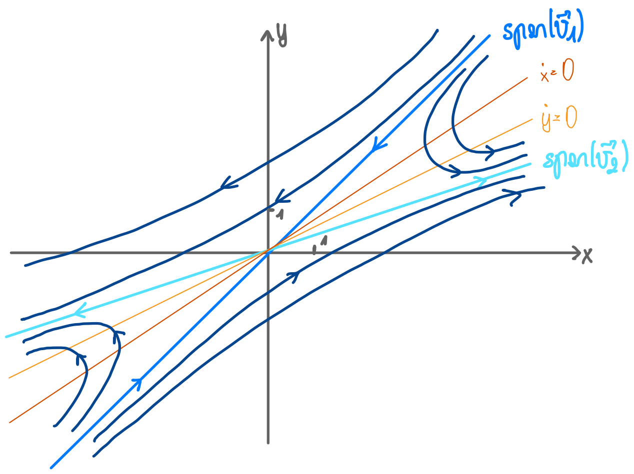


$z_1(t)=z_1(0)e^{-2t}$
$z_2(t)=z_2(0)e^{2t}$

$\Rightarrow z_2(t)=z_2(0)e^{2t}=z_1(0)z_2(0)\frac{1}{z_1(t)}$

$\rightsquigarrow$ OF THE FORM: $z_2=a\frac{1}{z_1}$ $(a\in\mathbb{R}, z_1\neq 0)$

·) PHASE DIAGRAM FOR $\dot{\vec{x}}=A\vec{x}$ WHERE $\vec{x}=\begin{pmatrix}x\\y\end{pmatrix}$



ISOCLINES:

·) $\dot{x}=0 \Rightarrow y=\frac{2}{3}x$

·) $\dot{y}=0 \Rightarrow y=\frac{1}{2}x$

1b) GENERAL SOLUTION $\vec{x}(t)=c_1e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}+c_2e^{2t}\begin{pmatrix}3\\1\end{pmatrix}$ $(c_1,c_2\in\mathbb{R})$

$\begin{pmatrix}2e^2\\0\end{pmatrix}=x(1)=c_1e^{-2}\begin{pmatrix}1\\1\end{pmatrix}+c_2e^{2}\begin{pmatrix}3\\1\end{pmatrix} \Rightarrow \left.\begin{matrix}c_1e^{-2}+3c_2e^2=2e^2\\ c_1e^{-2}+c_2e^2=0\end{matrix}\right\}- \Rightarrow \begin{matrix}2c_2e^2=2e^2\\ c_1e^{-2}+c_2e^2=0\end{matrix} \Rightarrow \begin{matrix}c_2=1\\ c_1=-e^4\end{matrix}$

$\Rightarrow x(t)=-e^{2(2-t)}\begin{pmatrix}1\\1\end{pmatrix}+e^{2t}\begin{pmatrix}3\\1\end{pmatrix}$ IS THE SOL WHICH SATISFIES: $\vec{x}(1)=\begin{pmatrix}2e^2\\0\end{pmatrix}$

2a) EP: $\begin{matrix}\dot{x}=0\\ \dot{y}=0\end{matrix} \Rightarrow \begin{matrix}y=x\\ y=x^2-2\end{matrix} \Rightarrow \begin{matrix}y=x\\ x^2-x-2=0\end{matrix} \Rightarrow \begin{matrix}y=x\\ x_{12}=\frac{1}{2}\pm\sqrt{\frac{1}{4}+2}=\frac{1}{2}\pm\frac{3}{2}\end{matrix}$

$\Rightarrow$ TWO EP: $(-1,-1)$, $(2,2)$

JACOBIAN $Df(\vec{x})=\begin{pmatrix}-1 & 1\\ -2x & 1\end{pmatrix}$ [$f(\vec{x})$... RHS OF THE SYSTEM, IE $\dot{\vec{x}}=f(\vec{x})$]

·) $Df((-1,-1))=\begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}$: EIGENVALUES: $\det(Df((-1,-1))-\lambda I)=0$

$(-1-\lambda)(1-\lambda)-2=0$

$\lambda^2-1-2=0$

$\lambda^2=3$

$\lambda_{12}=\pm\sqrt{3}$

$\Rightarrow (-1,-1)$ IS A SADDLE POINT

·) $Df((2,2))=\begin{pmatrix}-1 & 1\\ -4 & 1\end{pmatrix}$: EIGENVALUES: $\det(Df((2,2))-\lambda I)=0$

$(-1-\lambda)(1-\lambda)+4=0$

$\lambda^2-1+4=0$

$\lambda^2=-3$

$\lambda_{12}=\pm\sqrt{3}i$

$\Rightarrow (2,2)$ IS A CENTRE, IF THE SYSTEM IS HAMILTONIAN.

$\dot{\vec{x}}=f(\vec{x})$ IS HAMILTONIAN IF THERE EXISTS $H(\vec{x})=H(x,y)$ ST $\begin{matrix}\dot{x}=H_y(x,y)\\ \dot{y}=-H_x(x,y)\end{matrix}$

$H_y(x,y)=y-x \Rightarrow H(x,y)=\frac{y^2}{2}-xy+C(x)$ [$C(x)$... FCT ONLY DEP ON x]

$\rightsquigarrow$

$H_x(x,y)=x^2-y-2 \Rightarrow H(x,y)=\frac{x^3}{3}-xy-2x+D(y)$ [$D(y)$... FCT ONLY DEP ON y]

$\Rightarrow H(x,y)=\frac{y^2}{2}-(y+2)x+\frac{x^3}{3}$ SATISFIES $\begin{matrix}\dot{x}=H_y(x,y)\\ \dot{y}=-H_x(x,y)\end{matrix}$

$\Rightarrow \dot{\vec{x}}=f(\vec{x})$ IS HAMILTONIAN $\Rightarrow (2,2)$ IS A CENTRE.

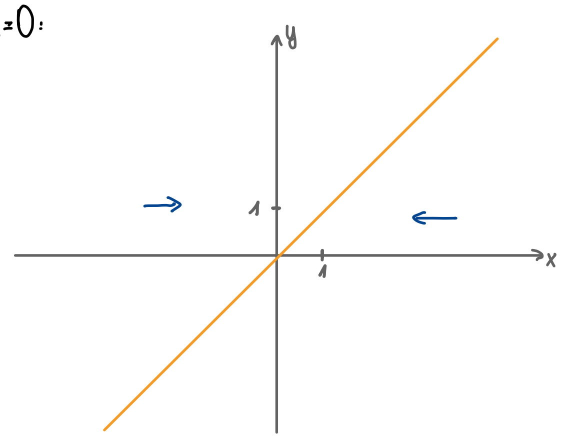
2b) FROM a): EP: $(-1,-1)$ ... SADDLE
$(2,2)$ ... CENTRE

·) ISOCLINES / NULLCLINES

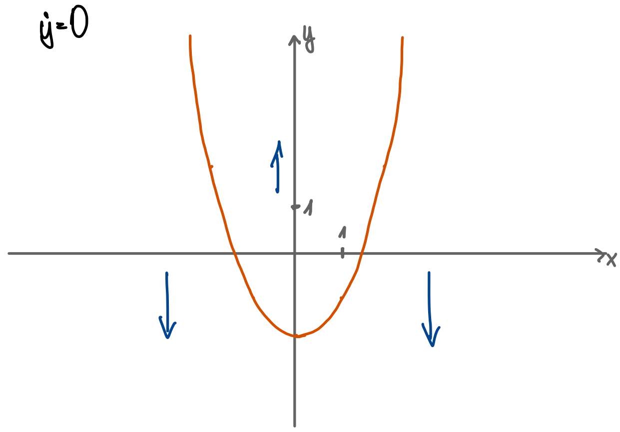
$\dot{x}=0 \Rightarrow y=x$

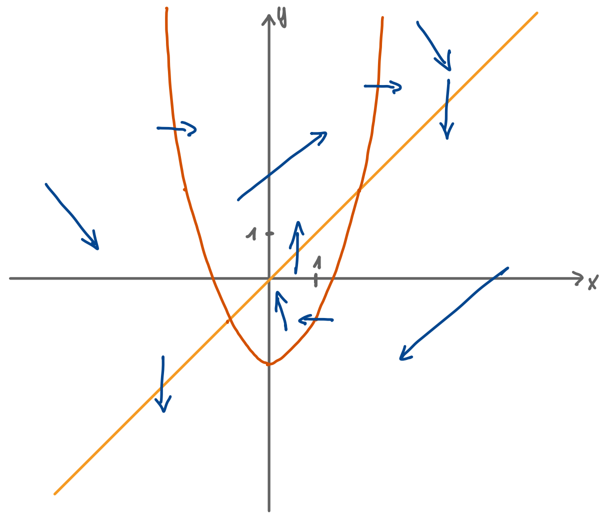
$\dot{y}=0 \Rightarrow y=x^2-2$

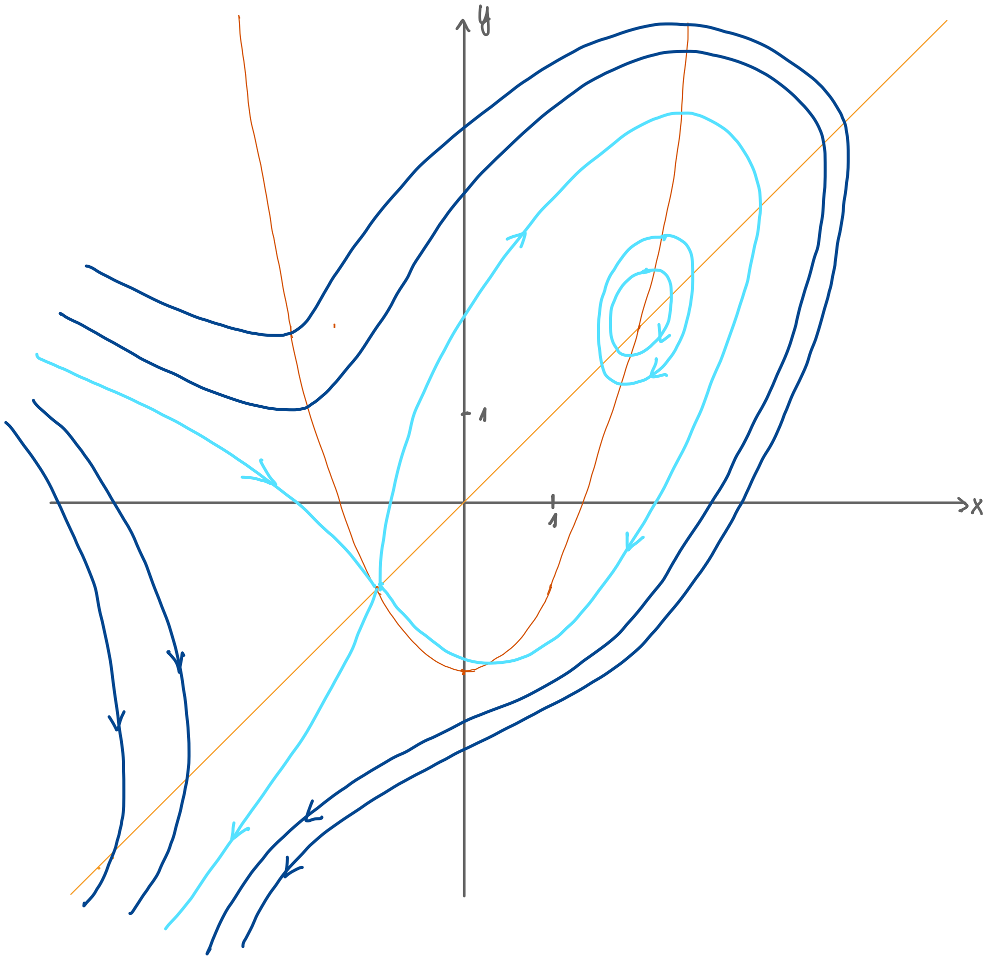
$\dot{x}=0$:



$\dot{y}=0$







3a) $\dot{x} = x^3 g(x,y) - y$
$\dot{y} = y^3 g(x,y) + x$

·) LINEARISATION AT (0,0)

$$\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x} = A\vec{x}$$ EIGENVALUES OF A: $\pm i$ ⇝ CAN'T USE LINEARISATION

·) LYAPUNOV FCT:

LET $V(x,y) = x^2 + y^2$ .... CONTINUOUS, STRICT MIN AT (0,0), $V(0,0) = 0$

$\dot{V}(x,y) = 2x\dot{x} + 2y\dot{y} = 2x(x^3 g(x,y) - y) + 2y(y^3 g(x,y) + x) = 2(x^4 + y^4) g(x,y) > 0$
$\forall \vec{x} = (x,y)$ ST $0 < |\vec{x}| \le 1$

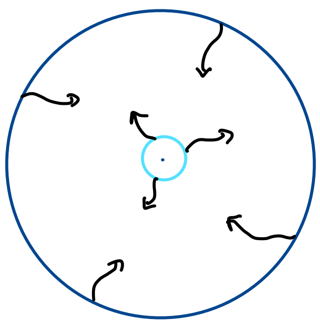
[LET $\vec{x} = (x,y)$ ST $|\vec{x}| \le 1 \Rightarrow g(x,y) = 12 + 4y - x^2 - y^2 \ge 12 - 4 - 1 = 7 > 0$]

$\dot{V}(0,0) = 0$

$\Rightarrow (0,0)$... UNSTABLE

3b) LET $\vec{x} = (x,y)$ ST $|\vec{x}| = 7 \Rightarrow g(x,y) = 12 + 4y - x^2 - y^2 \le 12 + 28 - 49 = -9 < 0$
$\Rightarrow \dot{V}(x,y) = 2(x^4 + y^4) g(x,y) < 0$

FROM a) $\vec{x} = (x,y)$ ST $|\vec{x}| = 1 \Rightarrow \dot{V}(x,y) = 2(x^4 + y^4) g(x,y) > 0$

⇒  ⇒ IF $\vec{x}(0) = x_0 \in B = \{\vec{x} \in \mathbb{R}^2 \mid 1 \le |\vec{x}| \le 7\}$,
THEN $\vec{x}(t) \in B$ $\forall t \ge 0$.

— ... $\{\vec{x} \in \mathbb{R}^2 \mid |\vec{x}| = 1\}$
— ... $\{\vec{x} \in \mathbb{R}^2 \mid |\vec{x}| = 7\}$

⇒ (POINCARÉ-BENDIXSON) THERE EXISTS A PERIODIC ORBIT IN B, IF B CONTAINS NO EP

·) EP: $\left.\begin{array}{l} \dot{x} = 0 \\ \dot{y} = 0 \end{array}\right. \Rightarrow \left.\begin{array}{l} y = x^3 g(x,y) \\ x = -y^3 g(x,y) \end{array}\right\} \Rightarrow \begin{array}{l} y = -y^9 g^4(x,y) \\ x = -y^3 g(x,y) \end{array} \Rightarrow \begin{array}{l} y(1 + y^8 g^4(x,y)) = 0 \\ x = -y^3 g(x,y) \end{array}$ ⇒ ONLY ONE EP: $(0,0) \notin B$

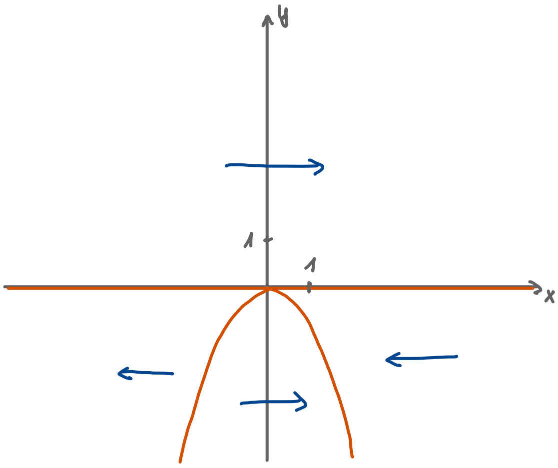
⇒ THERE EXISTS A PERIODIC ORBIT INSIDE B

4) EP: $\begin{matrix}\dot{x}=0\\ \dot{y}=0\end{matrix} \Rightarrow \begin{matrix} y(x^2+y)=0 \\ x+y^2=0 \end{matrix} \Rightarrow \left.\begin{matrix} y=0 \text{ OR } y=-x^2 \\ x=-y^2 \end{matrix}\right\} \Rightarrow (0,0) \text{ OR } \begin{matrix} y=-x^2 \\ x=-y^2=-x^4 \end{matrix} \Rightarrow (0,0) \text{ OR } \begin{matrix} y=-x^2 \\ x(x^3+1)=0 \end{matrix}$

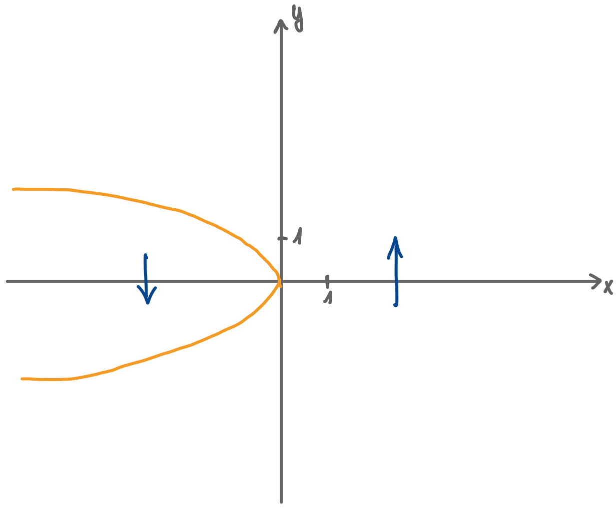
$\Rightarrow$ TWO EP $(0,0)$ AND $(-1,-1)$

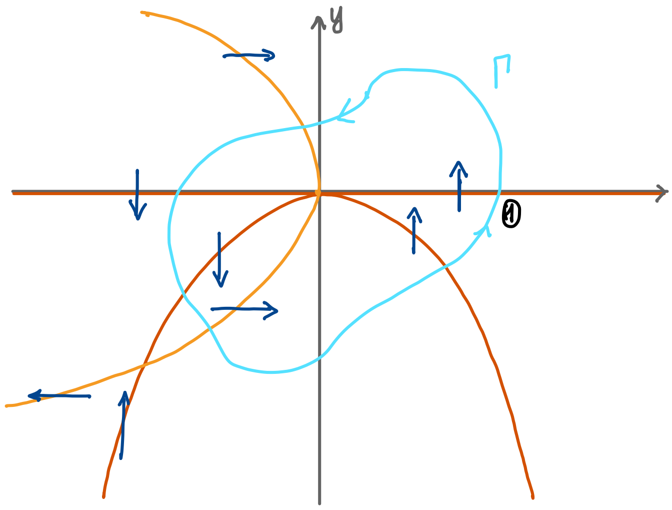
ISOCLINES / NULLCLINES:

$\dot{x}=0$



$\dot{y}=0$





LET $f(x,y)=\begin{pmatrix} y(x^2+y) \\ x+y^2 \end{pmatrix}$

INDEX:

$I_f((0,0))=I_f(\Gamma)=0$

$\uparrow \rightsquigarrow \rightarrow \rightsquigarrow \downarrow \rightsquigarrow \downarrow \rightsquigarrow \rightarrow \rightsquigarrow \uparrow \rightsquigarrow \uparrow$

$\rightsquigarrow$ 0 TURNS

5a) LET $\varepsilon \in \mathbb{R}$ AND ASSUME WLOG $x<y$:

$$|f^\varepsilon(y)-f^\varepsilon(x)| = \left|\cos\left(\tfrac{\pi}{1+\varepsilon^2}y\right)-\cos\left(\tfrac{\pi}{1+\varepsilon^2}x\right)\right| = \left|\int_{\frac{\pi}{1+\varepsilon^2}x}^{\frac{\pi}{1+\varepsilon^2}y} -\sin(z)\,dz\right| \le \int_{\frac{\pi}{1+\varepsilon^2}x}^{\frac{\pi}{1+\varepsilon^2}y} |\sin(z)|\,dz \le \frac{\pi}{1+\varepsilon^2}(y-x) = \frac{\pi}{1+\varepsilon^2}|y-x|$$

$\Rightarrow f^\varepsilon$ IS GLOBALLY LIPSCHITZ CONTINUOUS WITH LIPSCHITZ CONSTANT $L=\frac{\pi}{1+\varepsilon^2}$

LET $\varepsilon=0 \Rightarrow \dot{x}=\cos(\pi x)$ HAS EP $x_n=\frac{1}{2}+n$ $(n\in\mathbb{Z})$ AND $\cos(\pi x)$ IS GLOBALLY LIPSCHITZ

$\Rightarrow$ (1) HAS A UNIQUE GLOBAL SOLUTION

$-\frac{1}{2}=x_{-1}<x(0)=\frac{1}{4}<x_0=\frac{1}{2} \Rightarrow -\frac{1}{2}<x(t)<\frac{1}{2} \ \forall t$

5b) $x^0(t) = \frac{1}{4} + \int_0^t f^0(x^0(s))\,ds$ AND $x^\varepsilon(s) = \frac{1}{4} + \int_0^t f^\varepsilon(x^\varepsilon(s))\,ds$

$$\Rightarrow |x^0(t) - x^\varepsilon(t)| = \left|\int_0^t f^0(x^0(s)) - f^\varepsilon(x^\varepsilon(s))\,ds\right|$$

$$\leq \int_0^t |f^0(x^0(s)) - f^\varepsilon(x^\varepsilon(s))|\,ds$$

$$\leq \int_0^t |f^0(x^0(s)) - f^\varepsilon(x^0(s))| + |f^\varepsilon(x^0(s)) - f^\varepsilon(x^\varepsilon(s))|\,ds$$

$$= \int_0^t |f^0(x^0(s)) - f^0\left(\frac{1}{1+\varepsilon^2}x^0(s)\right)|\,ds + \int_0^t |f^\varepsilon(x^0(s)) - f^\varepsilon(x^\varepsilon(s))|\,ds$$

$$\overset{a)}{\leq} \int_0^t \pi\left|1 - \frac{1}{1+\varepsilon^2}\right||x^0(s)|\,ds + \int_0^t \pi\frac{1}{1+\varepsilon^2}|x^0(s) - x^\varepsilon(s)|\,ds \qquad \text{(LIPSCHITZ CONST)}$$

$$\overset{a)}{\leq} \pi\frac{\varepsilon^2}{1+\varepsilon^2}\frac{1}{2}t + \pi\frac{1}{1+\varepsilon^2}\int_0^t |x^0(s) - x^\varepsilon(s)|\,ds \qquad (x_0(s) \text{... BOUNDED})$$

$$\Rightarrow \left(e^{-\frac{\pi}{1+\varepsilon^2}t}\int_0^t |x^0(s) - x^\varepsilon(s)|\,ds\right)_t \leq \pi\frac{\varepsilon^2}{1+\varepsilon^2}\frac{1}{2}te^{-\frac{\pi}{1+\varepsilon^2}t}$$

$$\Rightarrow e^{-\frac{\pi}{1+\varepsilon^2}t}\int_0^t |x^0(s) - x^\varepsilon(s)|\,ds \leq \frac{\pi}{2}\frac{\varepsilon^2}{1+\varepsilon^2}\int_0^t se^{-\frac{\pi}{1+\varepsilon^2}s}\,ds$$

$$= -\frac{\varepsilon^2}{2}te^{-\frac{\pi}{1+\varepsilon^2}t} + \frac{\varepsilon^2}{2}\int_0^t e^{-\frac{\pi}{1+\varepsilon^2}s}\,ds$$

$$= -\frac{\varepsilon^2}{2}te^{-\frac{\pi}{1+\varepsilon^2}t} - \frac{\varepsilon^2(1+\varepsilon^2)}{2\pi}\left(e^{-\frac{\pi}{1+\varepsilon^2}t} - 1\right)$$

$$\Rightarrow \int_0^t |x^0(s) - x^\varepsilon(s)|\,ds \leq -\frac{\varepsilon^2}{2}t + \frac{\varepsilon^2(1+\varepsilon^2)}{2\pi}\left(e^{\frac{\pi}{1+\varepsilon^2}t} - 1\right)$$

$$\Rightarrow |x^0(t) - x^\varepsilon(t)| \leq \frac{\pi}{2}\frac{\varepsilon^2}{1+\varepsilon^2}t + \frac{\pi}{1+\varepsilon^2}\int_0^t |x^0(s) - x^\varepsilon(s)|\,ds \leq \frac{1}{2}(e^{\pi t} - 1)\,\varepsilon^2 = K(t)\,\varepsilon^2$$

7) $\Sigma = \{(x,0) \in \mathbb{R}^2 \mid x > 0\}$: $\vec{x} \in \Sigma \Rightarrow \vec{x} = (r\cos(\theta), r\sin(\theta)) = (r,0)$ WHERE $r = |\vec{x}|$ AND $\theta = 2n\pi$ $(n \in \mathbb{Z})$

$\Rightarrow$ IF $\vec{x}(0) = \vec{x}_0 \in \Sigma$, THEN $\vec{x}(t, x_0)$ WILL LEAVE Σ BUT RETURN TO SIGMA IF θ HAS CHANGED BY 2π, SINCE $\dot{\theta} = 2 > 0$.

$\Rightarrow$ THE TIME $\tau(\vec{x}_0)$ AT WHICH $\vec{x}(t, \vec{x}_0)$ RETURNS TO Σ FOR THE FIRST TIME IS GIVEN BY

$$\theta(T) - \theta(0) = 2\tau(\vec{x}_0) = 2\pi$$

$$\Rightarrow \tau(\vec{x}_0) = \pi$$

$\Rightarrow \Pi(\vec{x}_0) = x(\tau(\vec{x}_0), x_0) = x(\pi, x_0)$

$\Rightarrow$ $(\vec{x}_0 = (r_0, 0))$: $\Pi((r_0,0)) = \Pi(\vec{x}_0) = \vec{x}(\pi, \vec{x}_0) = (r(\pi), 0)$ WHERE $r(t)$ IS THE SOL TO (2a)

$$\Rightarrow \Pi((r_0,0)) = (r(\pi), 0) = \left(\frac{r(0)}{e^{-\pi} + r(0)(1 - e^{-\pi})}, 0 \right)$$

$$\Rightarrow \Pi^n((r_0,0)) = (r(n\pi), 0) = \left(\frac{r(0)}{e^{-n\pi} + r(0)(1 - e^{-n\pi})}, 0 \right) \longrightarrow (1,0) \quad (n \to \infty)$$

$\Rightarrow \Pi^n(\vec{x}_0) = \vec{x}(n\pi, \vec{x}_0) \to (1,0) \quad \forall \vec{x}_0 \in \Sigma$

$\Rightarrow (1,0) \in \omega(\Gamma_{\vec{x}_0}) \quad \forall \vec{x}_0 \in \Sigma \qquad$ ($\Gamma_{\vec{x}_0}$ ... ORBIT THROUGH $\vec{x}_0$)

$\Rightarrow \omega(\Gamma_{\vec{x}_0}) = \mathcal{C}$ ... UNIT CIRCLE $\forall \vec{x}_0 \in \Sigma$, SINCE $\Gamma_{(1,0)} = \mathcal{C}$ AND $\omega(\Gamma_{\vec{x}_0})$ IS CONNECTED

$\Rightarrow \Gamma_{(1,0)}$ IS ASYMPTOTICALLY STABLE