Homewor #1 TMA 4170, 2014

1 Let f be 2π-periodic function, and

$$\int_{-\pi}^{\pi} |f(t)|\, dt < \infty .$$

Let $f(t) \sim \sum c_n(f)\, e^{int}$ be its Fourier series

a) Fill the empty spaces in the table below:

f is real	$\Longleftrightarrow$	$c_{-n} = \overline{c_n}$
f is even	$\Longleftrightarrow$	?
?	$\Longleftrightarrow$	$c_{-n} = -c_n$
?	$\Longleftrightarrow$	c_n is pure imaginary and $c_n = -c_{-n}$

b) Fix $t_0 \in \mathbb{R}$. What is the Fourier series of $f(t - t_0)$

2. Find the Fourier series of $\sin^3 t$.
Can you do this without evaluating integrals?

3. Fix $x \in \mathbb{R}$ and consider the 2π-periodic function $f(t) = e^{x e^{it}}$. Find the Fourier series of f.

4. Denote by ℓ^1 the space of all sequences $\{c_n\}_{n=-\infty}^{\infty}$ such that $\sum |c_n| < \infty$ and by ℓ^2 the space of all sequences $\{c_n\}_{n=-\infty}^{\infty}$ such that $\sum |c_n|^2 < \infty$.

Prove that $\ell^1 \subset \ell^2$ and give an example of sequence $\{c_n\} \in \ell^2$, $\{c_n\} \notin \ell^1$.

Remark: We had opposite for $L^1(-\pi,\pi)$ and $L^2(-\pi,\pi)$. Why?

5. Let $f(x) = \cos ax$, $-\pi \le x < \pi$ where a is not an integer. Find Fourier series for $f(x)$. Use this expansion in order to prove that

$$\frac{1}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z - n\pi} + \frac{1}{z - n\pi} \right] \qquad (*)$$

where z is any complex number which is not a multiple of π

Additional exercise for those who attends Complex analysis:

Obtain expression $(*)$ without Fourier series by using the Cauchy residues theorem.