Item (d) justifies calling P *the* projection upon A. Now check that A inherits the separability of $L^2(Q)$.

EXERCISE 14. Show that the family f_n: $n \geqslant 1$ spans $L^2(Q)$ iff $(f,f_n)=0$ for every $n \geqslant 1$ implies $f \equiv 0$. *Hint:* What is the annihilator of the family f_n: $n \geqslant 1$?

EXERCISE 15. Show that any linear map l of $L^2(Q)$ into the complex numbers which is bounded in the sense that

$$|l(f)| \leqslant \text{constant} \times \|f\|$$

where the constant is independent of f, can be expressed as an inner product:

$$l(f) = (f,g)$$

for some $g \in L^2(Q)$. This is the so-called *Riesz representation theorem*. *Hint.* Suppose $l \not\equiv 0$ and let $A = (f: l(f)=0)$. Check that $B = A^0$ is of dimension 1 and find a function $g \in L^2(Q)$ so that $l(f)=0$ iff $(f,g)=0$. Then

$$l(f) = \|g\|^{-2} l(g)\,(f,g) = \text{constant} \times (f,g).$$

Why?

There are many excellent books on Hilbert space; among the best at an advanced level are Akhiezer and Glazman [1961–1963] and Riesz and Sz.-Nagy [1955]; at an elementary level, Berberian [1961] is recommended.

1.4 SQUARE SUMMABLE FUNCTIONS ON THE CIRCLE AND THEIR FOURIER SERIES

Attention is now focused on the space $L^2(S^1)$, where S^1 is a unit circular circumference. This may be thought of as the interval $0 \leqslant x \leqslant 1$ with the endpoints 0 and 1 identified, and you may think of functions on S^1 as periodic functions on R^1 of period 1, so that $f(x+1)=f(x)$. A function f is continuous on the circle only if $f(0)=f(0+)=f(1)=f(1-)$. You can also picture the circle as the interval $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$ with identification of endpoints; this attitude will be helpful occasionally. The space $L^2(S^1)$ is the Hilbert space of (complex) measurable functions f on S^1 with

$$\|f\| = \|f\|_2 = \left(\int_0^1 |f|^2\right)^{1/2} < \infty.$$

The inner product is

$$(f,g) = \int_0^1 fg^*,$$

and there is a self-evident isomorphism with $L^2[0,1]$. The space $C^n(S^1)$ of $n\,(<\infty)$ times continuously differentiable functions on the circle will also come into play, as well as the space $C^\infty(S^1)$ of infinitely differentiable functions. The space $C^\infty(S^1)$ is dense in $L^2(S^1)$ by Exercise 1.2.11. The space $C^0(S^1) = C(S^1)$ is just the space of continuous functions; for such functions, the new length

$$\|f\|_\infty = \max_{0\leqslant x<1} |f(x)|$$

is often convenient. The subscript ∞ will distinguish this notion of length from

$$\|f\|_2 = \left(\int_0^1 |f|^2\right)^{1/2} \leqslant \|f\|_\infty .$$

THEOREM 1. *The unit-perpendicular family*

$$e_n(x) = e^{2\pi inx}, \qquad n \in Z^1$$

of Example 1.3.1 *is a basis for* $L^2(S^1)$, *that is, any function* $f \in L^2(S^1)$ *can be expanded into a Fourier series*

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e_n$$

with coefficients

$$\hat{f}(n) = (f, e_n) = \int_0^1 f e_n^* = \int_0^1 f(x)\, e^{-2\pi inx}\, dx,$$

the sum being understood in the sense of distance in $L^2(S^1)$. *By Theorem* 1.3.3, *the map* $f \to \hat{f}$ *is therefore an isomorphism of* $L^2(S^1)$ *onto* $L^2(Z^1)$ [*see Exercise* 1.2.13], *and there is a Plancherel identity:*

$$\|f\|_2^2 = \int_0^1 |f|^2 = \|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 .$$

Warning: Until further notice e_n *will always mean* $e^{2\pi inx}$; *and "Fourier series" will refer to this particular family*!

EXERCISE 1. $1, \sqrt{2}\cos 2\pi nx : n \geqslant 1$, and $\sqrt{2}\sin 2\pi nx : n \geqslant 1$ form a unit-perpendicular family in $L^2(S^1)$. Check this and deduce that the (complex) Fourier series for f can also be expressed in the real form

$$f = \hat{f}_{\text{even}}(0) + \sum_{n=1}^{\infty} [\hat{f}_{\text{even}}(n)\sqrt{2}\cos 2\pi nx + \hat{f}_{\text{odd}}(n)\sqrt{2}\sin 2\pi nx],$$

with coefficients

$$\hat{f}_{\text{even}}(0) = \int_0^1 f(x)\,dx\,,$$

$$\hat{f}_{\text{even}}(n) = \sqrt{2}\int_0^1 f(x)\cos 2\pi nx\,dx\,, \qquad n \geqslant 1\,,$$

$$\hat{f}_{\text{odd}}(n) = \sqrt{2}\int_0^1 f(x)\sin 2\pi nx\,dx\,, \qquad n \geqslant 1\,.$$

The key step in proving that the exponentials $e_n : n \in Z^1$ span $L^2(S^1)$ is to check that the Fourier series of a smooth function f actually converges (to f!). This is the content of the following theorem.

THEOREM 2. *For any* $1 \leqslant p < \infty$ *and any* $f \in C^p(S^1)$, *the partial sums*

$$S_n = S_n(f) = \sum_{|k| \leqslant n} \hat{f}(k) e_k$$

converge to f, *uniformly as* $n \uparrow \infty$; *in fact,* $\|S_n - f\|_\infty$ *is bounded by a constant multiple of* $n^{-p+\frac{1}{2}}$.

Amplification: The bound on $\|S_n - f\|_\infty$ indicates that the speed of convergence of a Fourier series improves with the smoothness of f. This reflects the fact that *local* features of f (such as smoothness) are reflected in *global* features of $\hat{f}$ (such as rapid decay at $n = \pm\infty$). *This local–global duality is one of the major themes of Fourier series and integrals, as you will see later.*

PROOF (for $p = 1$). Bring in the so-called *Dirichlet kernel*

$$\begin{aligned} D_n(x) = \sum_{|k| \leqslant n} e_k(x) &= \sum_{|k| \leqslant n} e^{2\pi ikx} \\ &= e^{-2\pi inx} \sum_{k=0}^{2n} e^{2\pi ikx} \\ &= e^{-2\pi inx} \frac{e^{2\pi i(2n+1)x} - 1}{e^{2\pi ix} - 1} \\ &= \frac{\sin \pi(2n+1)x}{\sin \pi x}\,, \end{aligned}$$

with the understanding that $D_n(0) = (2n+1)$, and note that you obtain the value of the forbidding-looking integral

$$\int_0^1 D_n = \int_0^1 \frac{\sin \pi(2n+1)x}{\sin \pi x}\,dx$$

as a fringe benefit:

$$\int_0^1 D_n = \sum_{|k|\leqslant n} \int_0^1 e_k = 1.$$

The introduction of D_n is motivated by the desire to express S_n in a more transparent way:

$$\begin{aligned} S_n(x) &= \sum_{|k|\leqslant n} \hat{f}(k)\, e_k(x) = \sum_{|k|\leqslant n} e_k(x) \int_0^1 f(y)\, e_k^*(y)\, dy \\ &= \int_0^1 \sum_{|k|\leqslant n} e_k(x-y) f(y)\, dy \\ &= \int_0^1 D_n(x-y) f(y)\, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+y)\, D_n(y)\, dy. \end{aligned}$$

To achieve the final expression, make the substitution $y-x \to y$, and use the fact that D_n is even:

$$\int_0^1 D_n(x-y) f(y)\, dy = \int_{-x}^{1-x} f(x+y)\, D_n(-y)\, dy = \int_{-x}^{1-x} f(x+y)\, D_n(y)\, dy;$$

then notice that $f(x+\cdot)D_n$ is a periodic function of period 1, so that you can replace the integral over $-x \leqslant y \leqslant 1-x$ by the integral over *any* period you like, for example, $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.

The problem of verifying that S_n is a good approximation to f can now be better understood with the aid of a picture of D_n; see Fig. 1 for a sketch of D_8. The peak tends to ∞ with n. At the same time, the oscillations to either side become increasingly rapid, and while they do not die away, you can

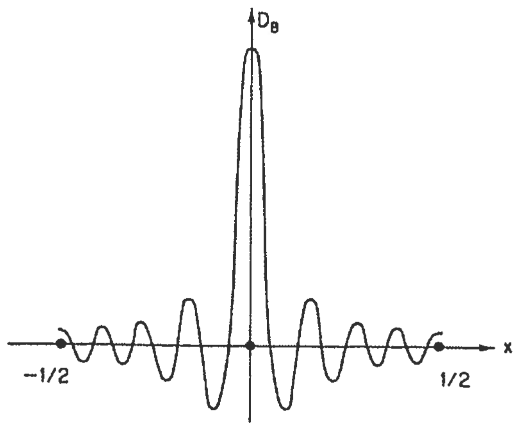

FIGURE 1

hope that they will, on the average, knock each other out, with the result that the major contribution to the integral comes from a small neighborhood of $y=0$. The plan is to take (indirect) advantage of this phenomenon.

To begin with, since $f \in C^1(S^1)$,

$$(f')^\wedge(n) = \int_0^1 f' e_n^* = -\int_0^1 f e_n'^* = 2\pi i n \hat{f}(n)$$

by partial integration. Therefore, by the inequalities of Schwarz and Bessel, if $n \leqslant n' < \infty$, then

$$\begin{aligned} |S_n - S_{n'}| &\leqslant \sum_{|k|>n} |\hat{f}(k)| = \sum_{|k|>n} |(f')^\wedge(k)|\, |2\pi k|^{-1} \\ &\leqslant \Big(\sum_{|k|>n} |(f')^\wedge(k)|^2\Big)^{1/2} \Big(\sum_{|k|>n} (2\pi k)^{-2}\Big)^{1/2} \\ &\leqslant \|f'\|_2 \times \text{a constant multiple of } n^{-1/2}. \end{aligned}$$

This shows that S_n converges uniformly, at the advertised speed, to *something*. The only question is: *to what*?

Because

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} D_n = 1,$$

the discrepancy between f and the partial sum can be expressed as

$$\begin{aligned} S_n(x) - f(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x+y) - f(x)]\, D_n(y)\, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} Q(x,y) \sin \pi(2n+1) y\, dy, \end{aligned}$$

with the understanding that

$$Q(x,y) = [f(x+y) - f(x)]/\sin \pi y$$

stands for $\pi^{-1} f'(x)$ at $y=0$. Fix $-\frac{1}{2} \leqslant x < \frac{1}{2}$. As a function of $-\frac{1}{2} \leqslant y < \frac{1}{2}$, Q belongs to $L^2[-\frac{1}{2}, \frac{1}{2})$, and

$$\begin{aligned} S_n(x) - f(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} Q(x,y) \frac{e^{2\pi i n y} e^{\pi i y} - e^{-2\pi i n y} e^{-\pi i y}}{2i}\, dy \\ &= (2i)^{-1} (Q^+)^\wedge(-n) - (2i)^{-1} (Q^-)^\wedge(n) \end{aligned}$$

in which $Q^\pm = Q \exp(\pm \pi i y)$. By a second application of Bessel's inequality

$$\sum |(Q^\pm)^\wedge(n)|^2 \leqslant \|Q\|_2^2 < \infty,$$

so $(Q^\pm)^\wedge(n)$ approaches 0 as $|n| \uparrow \infty$, and $\lim_{n\uparrow\infty} S_n = f$ for each fixed $-\frac{1}{2} \leqslant x < \frac{1}{2}$, separately. Now make $n' \uparrow \infty$ in the preceding estimate for $|S_n - S_{n'}|$ to verify that $\|S_n - f\|_\infty$ is bounded by a constant multiple of $n^{-1/2}$. This completes the proof for the case $p=1$.

EXERCISE 2. Finish the proof of Theorem 1. *Hint:* The space $C^1(S^1)$ is dense in $L^2(S^1)$ by Exercise 1.2.11.

EXERCISE 3. Finish the proof of Theorem 2 for $2 \leqslant p < \infty$. *Hint:* $(f')^\wedge(n) = 2\pi i n \hat{f}(n)$.

EXERCISE 4. Check that $f \in C^\infty(S^1)$ iff $\hat{f}$ is *rapidly decreasing* in the sense that $n^p \hat{f}(n)$ approaches 0 as $|n| \uparrow \infty$ for every $p < \infty$, separately. *Hint:* For rapidly decreasing $\hat{f}$, $\sum \hat{f}(n) e_n'$ converges uniformly to a periodic function f_1, and

$$\int_0^x f_1 = \sum \hat{f}(n) \int_0^x e_n' = \sum \hat{f}(n) \left[e_n(x) - e_n(0) \right] = f(x) - f(0).$$

The pointwise convergence of the Fourier series of a function from $L^2(S^1)$ is a very complicated business in general. Carleson [1966] proved that it must converge a.e., but there are examples with $f \in C(S^1)$ in which the sum diverges at uncountably many points. The situation for summable functions $[\int_0^1 |f| < \infty]$ is even more horrible; the most famous example is that of Kolmogorov [1926] in which the sum diverges everywhere! The proofs by Carleson and Kolmogorov are complicated and cannot be presented here; in fact, these remarks are merely meant to point out the very attractive simplicity of the following result of Fejér [1904].

THEOREM 3. *For functions f of class $C(S^1)$, the arithmetic means $n^{-1}(S_0 + \cdots + S_{n-1})$ of the partial sums $S_n = \sum_{|k| \leqslant n} \hat{f}(k) e_k$ converge uniformly to f.*

EXERCISE 5. Check that the arithmetic means $n^{-1}(x_0 + \cdots + x_{n-1})$ of the numerical series $x_0, x_1, \ldots$ converge to y if $\lim_{n\uparrow\infty} x_n = y$. Give an example to show that $\lim_{n\uparrow\infty} n^{-1}(x_0 + \cdots + x_{n-1})$ can exist even if $\lim_{n\uparrow\infty} x_n$ does not.

PROOF OF THEOREM 3. The proof makes use of the arithmetical means of the Dirichlet kernel:

$$\begin{aligned} F_n(x) &= \frac{1}{n}(D_0 + \cdots + D_{n-1}) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sin \pi(2k+1)x}{\sin \pi x} \\ &= \frac{1}{n} \left[\frac{\sin n\pi x}{\sin \pi x} \right]^2. \end{aligned}$$

EXERCISE 6. Check the summation. *Hint:* Think of $\sin \pi(2k+1)x$ as the imaginary part of the complex exponential and sum the resulting geometrical series.

F_n is the so-called *Fejér kernel.* Note for future use that F_n is nonnegative and that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n = n^{-1} \sum_{k=0}^{n-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} D_k = 1 .$$

The proposal is to check that the discrepancy

$$\begin{aligned} n^{-1} \sum_{k=0}^{n-1} S_k(x) - f(x) &= n^{-1} \sum_{k=0}^{n-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+y)\, D_k(y)\, dy - f(x) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+y)\, F_n(y)\, dy - f(x) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x+y) - f(x)]\, F_n(y)\, dy \end{aligned}$$

is uniformly small. This is much easier for the present Fejér kernel F_n than it was for the Dirichlet kernel D_n since now the tails are small for large n and not just negligible due to rapid oscillation; see Fig. 2 for a sketch of F_8.

To make the actual estimates pick a small positive number $\delta < \frac{1}{2}$ and divide the interval of integration into two parts according as $|y| < \delta$ or $|y| \geqslant \delta$. The first piece is bounded as follows:

$$\begin{aligned} \left| \int_{|y|<\delta} \right| &\leqslant \int_{-\delta}^{\delta} |f(x+y) - f(x)|\, F_n(y)\, dy \\ &\leqslant \max_{|y|\leqslant\delta} \max_{|x|\leqslant\frac{1}{2}} |f(x+y) - f(x)| \int_{|y|<\delta} F_n(y)\, dy \\ &\leqslant \max_{|y|\leqslant\delta} \|f_y - f\|_\infty , \end{aligned}$$

in which f_y is the translated function $f_y(x) = f(x+y)$. This can be made as small as you please by making $\delta \downarrow 0$, since f is uniformly continuous. As to the other piece,

$$\begin{aligned} \left| \int_{|y|\geqslant\delta} \right| &\leqslant \frac{4}{n} \|f\|_\infty \int_{\delta}^{\frac{1}{2}} \left[\frac{\sin n\pi x}{\sin \pi x} \right]^2 dx \\ &\leqslant \frac{2}{n} (\sin \pi\delta)^{-2} \|f\|_\infty , \end{aligned}$$

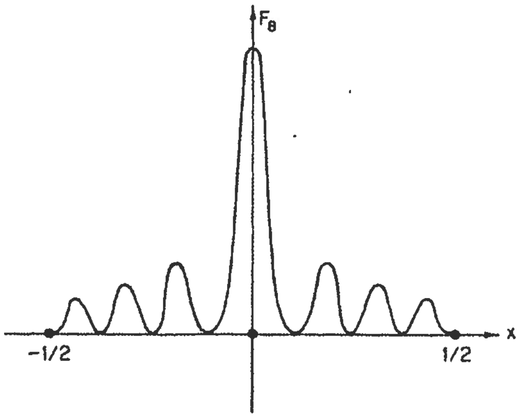

FIGURE 2

and this can be made small for fixed $0 < \delta < \frac{1}{2}$ by making n large enough. The proof is finished.

EXERCISE 7. Check that for any $f \in L^2(S^1)$

$$\lim_{n \uparrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) = \hat{f}(0) = \int_0^1 f$$

in the sense of distance in $L^2(S^1)$. *Hint:* Compute the Fourier coefficients of the sum and use the Plancherel identity.

EXERCISE 8. Check that for any $f \in C(S^1)$ and $0 \leqslant r < 1$,

$$\sum \hat{f}(n) r^{|n|} e_n(x) = \int_0^1 \frac{1 - r^2}{1 - 2r \cos 2\pi(x-y) + r^2} f(y)\, dy .$$

Hint: Express $\hat{f}(n)$ as an integral and bring the sum under the integral sign.

EXERCISE 9. Use Exercise 8 to prove the following variant of Theorem 3: For $f \in C(S^1)$,

$$\lim_{r \uparrow 1} \| \sum \hat{f}(n) r^{|n|} e_n - f \|_\infty = 0 .$$

The fact goes back to Poisson; the formula of Exercise 8 is known by his name. *Hint:*

$$\int_0^1 \frac{1-r^2}{1-2r\cos 2\pi y + r^2}\, dy = \sum \hat{1}(n) r^{|n|} e_n(0) = 1 .$$

1.5 SUMMABLE FUNCTIONS AND THEIR FOURIER SERIES

The next topic is the space $L^1(S^1)$ of (complex) summable functions on the circle. This space, or more generally, the space $L^1(Q)$ of summable functions on an interval Q, is provided with a length

$$\|f\|_1 = \int_Q |f| = \int_Q |f(x)|\, dx,$$

permitting you to define a distance much as for $L^2(Q)$. Clearly, the triangle inequality,

$$\|f+g\|_1 = \int_Q |f+g| \leqslant \int_Q |f| + \int_Q |g| = \|f\|_1 + \|g\|_1,$$

is satisfied, so that $L^1(Q)$ is closed under addition of functions, and it goes without saying that it is also closed under multiplication by complex numbers.

Technical point: The reader should bear in mind that, just as for $L^2(Q)$ the things that live in $L^1(Q)$ are actually *equivalence classes* of functions; especially, you identify f with the 0-function if $f=0$ a.e. so as to make $f=0$ iff $\|f\|_1 = 0$. As before, it is simpler, and not at all confusing, to ignore this point most of the time and to speak of $f \in L^1(Q)$ as a *function*.

EXERCISE 1. Check that the space $L^1(Q)$ is *not* a Hilbert space. *Hint:* In any Hilbert space $\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2$. Try this out for $Q = [0, 1]$, $\alpha = 1$, and $\beta = x$. What happens? Try again.

EXERCISE 2. Show that the space $L^1(Q)$ is *complete* and *separable*. The definitions are as for $L^2(Q)$, but relative to the distance of $L^1(Q)$. *Hint:* The proof for $L^2(Q)$ is easily adapted from Section 1.2 with minor changes only.

EXERCISE 3. Show that the class of compact functions belonging to $C^\infty(Q)$ is dense in $L^1(Q)$. *Hint:* See Exercise 1.2.11. There is an inclusion between $L^1(Q)$ and $L^2(Q)$ if Q is bounded: Namely, by Schwarz's inequality,

$$\|f\|_1^2 = \left(\int_Q |f| \times 1\right)^2 \leqslant \int_Q |f|^2 \int_Q 1^2 = \|f\|_2^2 m(Q),$$

so that $L^2(Q)$ is included in $L^1(Q)$. In particular, this holds for functions on the circle, so that everything to be proved later about Fourier series of summable functions on the circle also holds for functions of class $L^2(S^1)$.

EXERCISE 4. Check by example that the inclusion of $L^2[0, 1]$ in $L^1[0, 1]$ is *proper*, that is, find a summable function f with $\int_0^1 |f|^2 = \infty$.

EXERCISE 5. Check by examples that neither $L^1(R^1)$ nor $L^2(R^1)$ is included in the other.

EXERCISE 6. Show that any linear map l of $L^1(Q)$ into the complex numbers, subject to $|l(f)| \leq \text{constant} \times \|f\|_1$, with a constant independent of f, can be expressed as $l(f) = \int_Q fg^*$ for some bounded measurable function g. *Hint:* $L^2(Q) \subset L^1(Q)$ if Q is bounded. Now use Exercise 1.3.15 to find such a function $g \in L^2(Q)$ and check that $\int_b^a |g| \leq \text{constant} \times (b-a)$ for any interval $a \leq x \leq b$.

Given $f \in L^1(S^1)$, the function fe_n^* is summable, so you can set up a (formal) Fourier series for f by the customary recipe:

$$f = \sum \hat{f}(n) e_n$$

with coefficients

$$\hat{f}(n) = \int_0^1 fe_n^* = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

The principal fact about such Fourier series is contained in

THEOREM 1. *The arithmetical means $n^{-1}(S_0 + \cdots + S_{n-1})$ of the partial sums $S_n = \sum_{|k| \leq n} \hat{f}(k) e_k$ converge to f in the sense of distance in $L^1(S^1)$:*

$$\lim_{n \uparrow \infty} \|n^{-1}(S_0 + \cdots + S_{n-1}) - f\|_1 = 0;$$

in particular, the map $f \to \hat{f}$ is 1:1.

PROOF. The discrepancy between $n^{-1}(S_0 + \cdots + S_{n-1})$ and f may be expressed by means of Fejér's kernel F_n, as in the proof of Theorem 1.4.3:

$$n^{-1}(S_0 + \cdots + S_{n-1}) - f = \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x+y) - f(x)] F_n(y)\, dy,$$

and the length of the discrepancy can be bounded as follows:

$$\begin{aligned} \|n^{-1}(S_0 + \cdots + S_{n-1}) - f\|_1 &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x+y) - f(x)|\, F_n(y)\, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x+y) - f(x)|\, dx \right) F_n(y)\, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \|f_y - f\|_1\, F_n(y)\, dy, \end{aligned}$$

in which f_y is the translated function $f_y(x) = f(x+y)$. The rest of the proof runs parallel to that of Theorem 1.4.3. The only new ingredient is contained in

EXERCISE 7. Show that the map $f \to f_y$ is continuous in the distance of $L^1(S^1)$, i.e.,

$$\lim_{y \to 0} \|f_y - f\|_1 = 0.$$

Hint: $\|f_y - f\|_1 \leq \|f_y - f\|_\infty$, for f from $C(S^1)$ and the latter is dense in $L^1(S^1)$.

The Fourier coefficients of a summable function f do not satisfy

$$\sum |\hat{f}(n)|^2 < \infty;$$

that happens only if $f \in L^2(S^1)$, but they are *bounded*:

$$|\hat{f}(n)| = \left| \int_0^1 f e_n^* \right| \leq \int_0^1 |f| = \|f\|_1 .$$

This crude estimate is much improved upon by the so-called *Riemann–Lebesgue lemma*:

THEOREM 2. *The Fourier coefficient $\hat{f}(n)$ of any function $f \in L^1(S^1)$ tends to 0 as $|n| \uparrow \infty$.*

PROOF. $\hat{f}(n) = \int f e_n^*$ can also be expressed as

$$\hat{f}(n) = -\int_0^1 f(x) \exp[-2\pi i n(x - (2n)^{-1})] = -\int_0^1 f(x + (2n)^{-1}) e_n^*(x)\, dx,$$

since $e^{\pi i} = -1$. Now average the two expressions for $\hat{f}(n)$:

$$\hat{f}(n) = \tfrac{1}{2} \int_0^1 f(x) e_n^*(x)\, dx - \tfrac{1}{2} \int_0^1 f(x + (2n)^{-1}) e_n^*(x)\, dx$$

$$= \tfrac{1}{2} \int_0^1 [f(x) - f(x + (2n)^{-1})] e_n^*(x)\, dx$$

and estimate as follows:

$$|\hat{f}(n)| \leq \tfrac{1}{2} \int_0^1 |f(x) - f(x + (2n)^{-1})|\, dx = \tfrac{1}{2} \|f - f_{1/2n}\|_1 .$$

By Exercise 7, this approaches 0 as $|n| \uparrow \infty$. The proof is finished.

An important application of the Riemann–Lebesgue lemma is to verify that the *local* convergence of the Fourier series of a summable function depends only upon the *local* behavior of the function. This is the content of

THEOREM 3. *Take $f \in L^1(S^1)$ vanishing near $x = 0$. Then S_n approaches 0 as $n \uparrow \infty$, uniformly near $x = 0$.*

Amplification: Obviously, the point $x=0$ is in no way special; in fact, if f and g are summable functions and if $f=g$ near some fixed point x_0 of the circle, then their partial Fourier sums behave in the same way in the vicinity of x_0: Namely, $S_n(f)-S_n(g)=S_n(f-g)$ tends to 0 uniformly, near x_0.

PROOF OF THEOREM 3. Suppose that $f=0$ for $|x|\leqslant\delta$ and express the partial sum S_n by means of the Dirichlet kernel of Section 1.4:

$$S_n(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+y)\frac{\sin(2n+1)\pi y}{\sin \pi y}\,dy.$$

Because $f(x+y)=0$ if both $|x|$ and $|y|$ are $\leqslant \delta/2$, the functions

$$f^{\pm}(y) = \frac{f(x+y)e^{\pm i\pi y}}{2i\sin \pi y}$$

are summable if $|x|\leqslant\delta/2$, and therefore

$$\begin{aligned} S_n(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(x+y)}{\sin\pi y}\,\frac{e^{\pi iy}e^{*}_{-n}(y)-e^{-\pi iy}e_n^{*}(y)}{2i}\,dy \\ &= (f^{+})^{\wedge}(-n)-(f^{-})^{\wedge}(n) \end{aligned}$$

tends to 0 as $n\uparrow\infty$ for any fixed $|x|\leqslant\delta/2$, by the Riemann–Lebesgue lemma. At the same time, for any $|x_1|\leqslant\delta/2$ and $|x_2|\leqslant\delta/2$,

$$\begin{aligned} |S_n(x_2)-S_n(x_1)| &\leqslant \int_{\frac{1}{2}\geqslant|y|\geqslant\delta/2} |f(x_2+y)-f(x_1+y)|\left|\frac{\sin(2n+1)\pi y}{\sin\pi y}\right| dy \\ &\leqslant \left(\sin\frac{\pi\delta}{2}\right)^{-1} \|f_{x_2}-f_{x_1}\|_1 \\ &= \left(\sin\frac{\pi\delta}{2}\right)^{-1} \|f_y-f\|_1 \end{aligned}$$

for $y=x_2-x_1$. By Exercise 7, this makes S_n continuous for $|x|\leqslant\delta/2$, uniformly in n, and a moment's reflection will convince you that this forces the convergence of S_n for $|x|\leqslant\delta/2$ to be uniform. The proof is finished.

EXERCISE 8. Dini's test states that if $f\in L^1(S^1)$ and if for fixed $|x|\leqslant\frac{1}{2}$, the function $y^{-1}[f(x+y)-f(x)]$ is summable, then $\lim_{n\uparrow\infty} S_n(x)=f(x)$. Prove it. *Hint:* Use the Dirichlet kernel and the Riemann–Lebesgue lemma, as earlier.

A new feature of summable functions now comes into play: The space

$L^1(S^1)$ is an algebra under the multiplication defined by the "convolution" product

$$f \circ g = \int_0^1 f(x-y)g(y)\,dy.$$

For fixed $0 \leqslant x < 1$, the product $f(x-\cdot)g$ may be nonsummable, so it is necessary to check that $f \circ g$ makes sense. To do this with all the proper technical flourishes, look first at the plane integral

$$I = \int_0^1 \int_0^1 |f(x-y)g(y)|\,dx\,dy.$$

The integrand $|f(x-y)g(y)|$ is a nonnegative (plane) measurable function, so the integral makes sense $[I \leqslant \infty]$, and by Fubini's theorem, you can evaluate it as an iterated integral:

$$I = \int_0^1 |g(y)|\,dy \int_0^1 |f(x-y)|\,dx = \int_0^1 |g| \int_0^1 |f| = \|f\|_1 \|g\|_1 < \infty.$$

Fubini is used once more to conclude from

$$I = \int_0^1 dx \int_0^1 |f(x-y)g(y)|\,dy < \infty$$

that $f(x-\cdot)g$ is summable for almost every $0 \leqslant x < 1$. Thus, $f \circ g(x)$ is given by an honest Lebesgue integral for almost every $0 \leqslant x < 1$ and is itself a (periodic) summable function:

$$\|f \circ g\|_1 \leqslant I = \|f\|_1 \|g\|_1 < \infty.$$

This kind of finicky proof is not very interesting, but it is important to understand precisely what is involved.

EXERCISE 9. Check that the product $f \circ g$ is associative and commutative, putting in all the technical details.

EXERCISE 10. Check that $L^2(S^1)$ is an ideal in $L^1(S^1)$. This means that $f \circ g$ belongs to $L^2(S^1)$ as soon as one of the factors does.

EXERCISE 11. Check that the Dirichlet and Fejér formulas for partial sums can be expressed as

$$S_n = f \circ D_n$$

and

$$n^{-1}(S_0 + \cdots + S_{n-1}) = f \circ F_n.$$

A pretty interplay takes place between the product $f \circ g$ and Fourier series, stemming from

$$(f \circ g)^\wedge = \hat{f}\hat{g};$$

this simple formula plays a very important role in applications, as you will see later.

PROOF. By Fubini's theorem,

$$\begin{aligned}(f \circ g)^\wedge(n) &= \int_0^1 \left[\int_0^1 f(x-y)g(y)\,dy\right] e_n^*(x)\,dx \\ &= \int_0^1\int_0^1 f(x-y)\,e_n^*(x-y)\,g(y)\,e_n^*(y)\,dx\,dy \\ &= \int_0^1 g(y)\,e_n^*(y)\,dy \left[\int_0^1 f(x-y)\,e_n^*(x-y)\,dx\right] \\ &= \int_0^1 ge_n^* \int_0^1 fe_n^* \\ &= \hat{f}(n)\,\hat{g}(n).\end{aligned}$$

EXERCISE 12. Check that $L^1(S^1)$ does not have a multiplicative identity. *Hint:* A multiplicative identity e would satisfy $e \circ f = f$. Now look at $\hat{e}$ keeping the Riemann–Lebesgue lemma in mind.

EXERCISE 13. Define f^n to be the n-fold product $f \circ \cdots \circ f$ of a summable function f. Prove that

$$\lim_{n\uparrow\infty}(\|f^n\|_1)^{1/n} = \|\hat{f}\|_\infty = \max_{|n|<\infty} |\hat{f}(n)|,$$

under the extra assumption that f belongs to $L^2(S^1)$.[1] *Hint:* The fact that $\|\hat{f}\|_\infty$ does not exceed the left-hand side is self-evident. To finish the proof use

$$\|f^n\|_1 = \int f^n\{\exp[i\arg(f^n)]\}^* = \sum \hat{f}^n\{\exp[i\arg(f^n)]\}^{\wedge *}.$$

The map $f \to \hat{f}$ maps $L^2(S^1)$ *onto* $L^2(Z^1)$, and

$$\|f\|_2 = \|\hat{f}\|_2 = (\sum|\hat{f}|^2)^{1/2} < \infty.$$

The situation for $L^1(S^1)$ is very much more complicated. The information

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[1] For a proof of this formula without the extra assumption see, for example, Edwards [1967, article 11.4.14].

at hand about the class A of Fourier coefficients $\hat{f}$ of summable functions may be summarized as follows:

(a) A is populated by bounded functions $\hat{f}$:

$$\|\hat{f}\|_\infty = \lim_{n\uparrow\infty}(\|f^n\|_1)^{1/n} \leqslant \|f\|_1 .$$

(b) $\hat{f}=0$ iff $f=0$, i.e., $\wedge$ is a 1:1 map.

(c) $\hat{f}=0$ at $\pm\infty$, i.e., $\lim_{|n|\uparrow\infty}\hat{f}(n)=0$.

(d) A is an algebra: namely,

$$(\hat{f}\hat{g})(n) = \hat{f}(n)\hat{g}(n) = (f\circ g)^\wedge(n).$$

Unfortunately, (a) and (c) do not suffice to single out precisely which functions $\hat{f}$ arise as Fourier coefficients of summable functions. The situation is thus entirely different from $L^2(S^1)$ where the condition

$$\|\hat{f}\|_2 = (\sum|\hat{f}|^2)^{1/2} < \infty$$

is decisive. The best information currently available indicates that A does not have *any* neat description.

EXERCISE 14. Check that the class B of summable functions $\hat{f}$ is a subalgebra of A. The adjective "summable" means that $\|\hat{f}\|_1 = \sum|\hat{f}(n)| < \infty$.

EXERCISE 15. Check that $B^\vee = (f=\sum\hat{f}(n)e_n : \hat{f}\in B)$ is a subalgebra of $C(S^1)$ under the *ordinary* multiplication of functions. *Hint:* $(fg)^\wedge(n) = \sum\hat{f}(n-k)\hat{g}(k)$.

A celebrated theorem of Wiener [1933, p. 91] states that if $f\in B^\vee$ is root-free [$f\neq 0$], then also $1/f\in B^\vee$. A wide variety of fascinating and delicate results about A and B have been obtained since that date: the advanced student will find a nice account in Edwards [1967].

1.6* GIBBS' PHENOMENON

Thus far, the object has been to show how *well* Fourier series converge. Gibbs' phenomenon has to do with how poorly they converge in the vicinity of a jump of f. The statement is that in the vicinity of a simple jump of the function f, the partial sums S_n *always overshoot the mark by about* 9%. This fact was pointed out by Gibbs in a letter to *Nature* [1899]. (Actually Gibbs' phenomenon was first described by the British mathematician Wilbraham [1848]; see Carslaw [1925] for the history.) The function Gibbs considered