1. Gabor analysis:

a. Operators: $\alpha, \beta > 0$; $\quad T_\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ - shift

$\qquad M_\beta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ - modulation.

They are defined as follows:

$$T_\alpha : f \mapsto f(t-\alpha) \quad ; \quad M_\beta : f \mapsto e^{2i\pi\beta t} f(t)$$

Time-frequency shift:

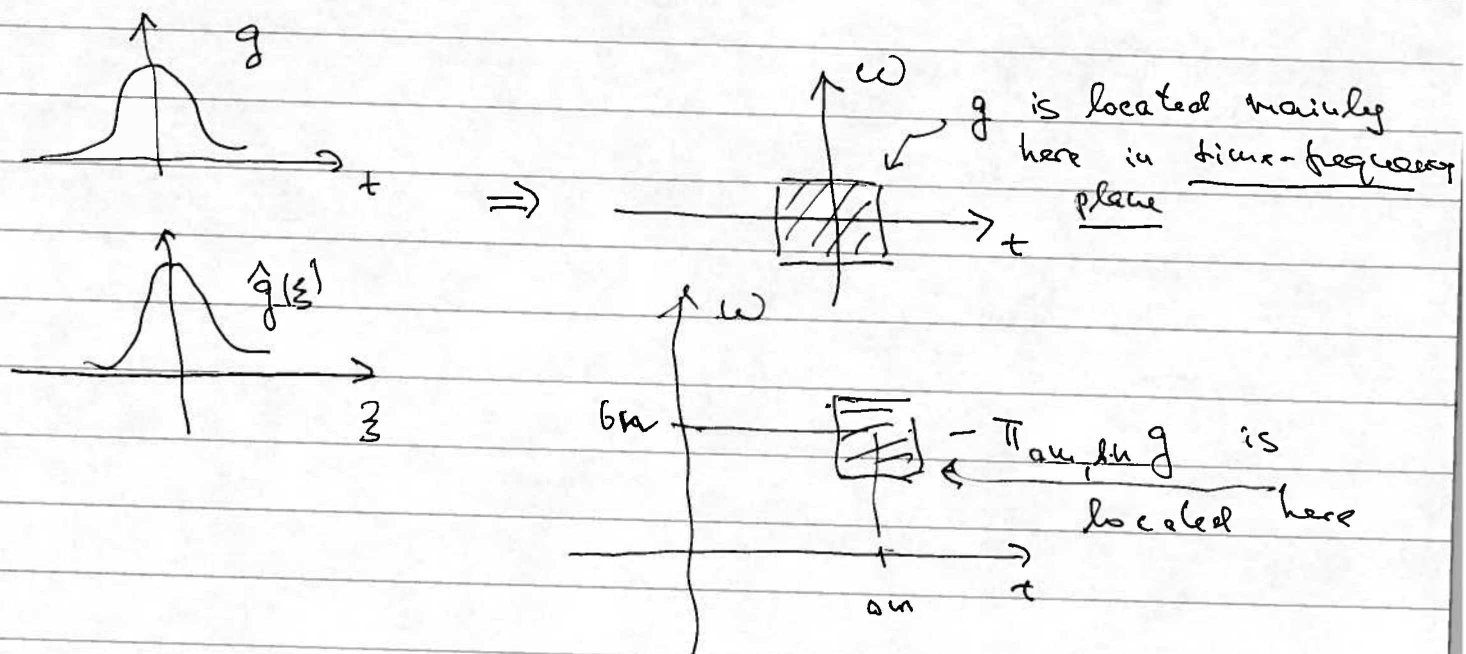
$$\pi_{\alpha\beta} f := M_\beta T_\alpha f = e^{2i\pi\beta t} f(t-\alpha)$$

b. Gabor system: Input: $g(t)$ - window, $a, b > 0$.

Output: Gabor system: $\boxed{\mathcal{G}(g;a,b)}$:

$$\mathcal{G}(g;a,b) = \{\pi_{an,bn}\, g\}_{m,n\in\mathbb{Z}} = \{e^{2i\pi bnt} g(t-an)\}_{m,n\in\mathbb{Z}}$$

c. Time-frequency interpretation:



d. Setting of the problem.

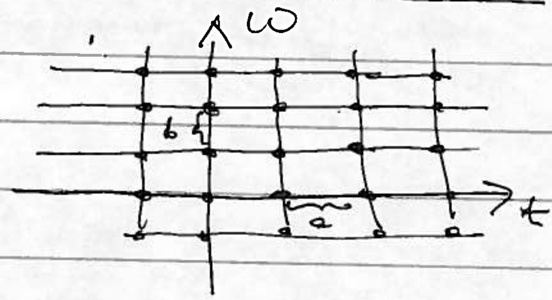
Given $f \in L^2(\mathbb{R})$ find representation

$$f(t) = \sum_{m,n} c_{mn} \, \Pi_{am,bn} \, g(t)$$

Interpretation: If $c_{m,n}$ is large then the signal carries frequency $\omega = bn$ at the moment $t = am$.

Heuristic: "Essential supports" of $\Pi_{am,bn}\, g$ should cover time-frequency plane sufficiently dense.

~~e. No go theorem: M. Rieffel: If a.~~



These supports are centered at .

Measure of density: $(ab)^{-1}$

e. No-go theorem

M. Rieffel: $ab > 1 \Rightarrow \mathcal{G}(g; a, b)$ does NOT span $L^2(\mathbb{R})$, ~~for any g~~ no matter which $g \in L^2(\mathbb{R})$ you take.

f. What happens if $ab \leq 1$?

Example: $g(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow \mathcal{G}(g; 1, 1)$ is an orthonormal basis in $L^2(\mathbb{R})$

Disadvantage: $\int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 \xi^2 \, d\xi = \infty$,

so it cannot be used for frequency localization

Exercise: Find another function g such that $\mathcal{G}(g, 1, 1)$ be an orthonormal basis in $L^2(\mathbb{R})$

g. Balian-Low theorem:

$g \in L^2(\mathbb{R})$, $\mathcal{G}(g; 1, 1)$ is an orthonormal basis in $L^2(\mathbb{R}) \Rightarrow$

$$\Rightarrow \int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt \cdot \int_{-\infty}^{\infty} \xi^2 |\hat{g}(\xi)|^2 \, d\xi = \infty .$$

h. Proof of BL theorem step 1

Coordinate and impuls operators:

$X : f \mapsto x f(x)$, $\quad P : f \mapsto \frac{1}{2i\pi} f'(x)$

~~Three~~ Properties:

- Unbounded
- Domains D_X, D_P, D_{XP}, D_{PX}
- $\left.\begin{array}{l}\langle Xf, g\rangle = \langle f, Xg\rangle \\ \langle Pf, g\rangle = \langle f, Pg\rangle\end{array}\right\}$ when well defined
- $(XP - PX)g = \frac{1}{2i\pi} g$ (!)

Digression: Quantum mechanical interpretation.

i. Idea of proof: We assume that

$$\int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt \int_{-\infty}^{\infty} \xi^2 |\hat{g}(\xi)|^2 \, d\xi < \infty$$

or equivalently $\boxed{g \in D}$: $Xg \in L^2$, $Pg \in L^2$.

If $\mathcal{G}(g; 1, 1)$ is an orthonormal basis then

$$Xg = \sum_{m,n} \langle Xg, \pi_{m,n} g\rangle \pi_{m,n} g$$

$$Pg = \sum_{m,n} \langle Pg, \pi_{m,n} g\rangle \pi_{m,n} g$$

and

$$\langle Xg, Pg\rangle = \sum_{m,n} \langle Xg, \pi_{m,n} g\rangle \langle \pi_{m,n} g, Pg\rangle$$

Direct calculation gives

$$\langle Xg, \pi_{m,n} g\rangle = \langle \pi_{-m,-n} g, Xg\rangle$$

$$\langle \pi_{m,n} g, Pg\rangle = \langle Pg, \pi_{-m,-n} g\rangle$$

and finally $\langle Xg, Pg\rangle = \langle Pg, Xg\rangle$

or $\langle PXg, g\rangle = \langle XPg, g\rangle$ or

$\langle (PX - XP)g, g\rangle = 0$ in contradiction with (!)

Remark: in this proof we assumed $g \in D_{PX} \cap D_{XP}$. One needs additional passage to the limit in the general case.