1.10 SEVERAL-DIMENSIONAL FOURIER SERIES

Consider the standard n-dimensional real number space R^n and the "lattice" $Z^n \subset R^n$ of points with integral coordinates. A function f on R^n is said to have "periods from Z^n" if it is periodic (of period 1) in each of its n variables:

$$f(x) = f(x+k) \qquad \text{for every} \quad k \in Z^n.$$

If you like, you can think of such a function as living on the standard n-dimensional torus

$$T^n\colon x = (x_1, \ldots, x_n), \qquad 0 \leqslant x_i < 1, \quad 1 \leqslant i \leqslant n.$$

The relation between R^n, Z^n, and T^n can be expressed as $T^n = R^n/Z^n$; for $n = 1$, this is simply the statement that the circle can be pictured as the real numbers mod 1.

1. Fourier Series on a Standard Torus

The space $L^2(T^n)$ is the set of measurable functions f on the standard torus T^n with

$$\|f\|^2 = \int_{T^n} |f(x)|^2 \, d^n x = \int_0^1 \cdots \int_0^1 |f(x_1, \ldots, x_n)|^2 \, dx_1 \ldots dx_n < \infty .$$

EXERCISE 1. Check that finite sums of products $f_1(x_1) \ldots f_n(x_n)$ of functions from $L^2(S^1)$ span $L^2(T^n)$.

By Exercise 1, the exponentials

$$e_k(x) = e^{2\pi i k \cdot x} = \exp[2\pi i(k_1 x_1 + \cdots + k_n x_n)] = e_{k_1}(x_1) \ldots e_{k_n}(x_n)$$

form a unit-perpendicular basis of $L^2(T^n)$ as $k = (k_1, \ldots, k_n)$ runs over the lattice of integral points Z^n. Any function $f \in L^2(T^n)$ can be expanded into an n-dimensional Fourier series

$$f = \sum_{k \in Z^n} \hat{f}(k) e_k$$

with coefficients

$$\hat{f}(k) = (f, e_k) = \int_{T^n} f e_k^* \, d^n x,$$

and there is a Plancherel formula

$$\|f\|^2 = \int_{T^n} |f|^2 = \|\hat{f}\|^2 = \sum_{Z^n} |\hat{f}(k)|^2 .$$

The extension of most of the one-dimensional results developed in Sections 1.4–1.8 is easy, although some small technical changes must be made. The following sample will suffice.

EXERCISE 2. The problem of heat flow on T^n is $\partial u/\partial t = \Delta u/2$ in which $\Delta = \partial^2/\partial x_1{}^2 + \cdots + \partial^2/\partial x_n{}^2$. Compute the solution and check that it tends to $\int f(x)\,d^n x$ as $t\uparrow\infty$ for any nice initial temperature f. *Hint:* Imitate Subsection 1.7.3, first for $n=2$ and then for general n.

2*. Application to Random Walks

Pólya [1921] discovered a very beautiful application of several-dimensional Fourier series to "random walks." Think of a particle moving on the d-dimensional lattice Z^d according to the following rule. The particle starts at time 0 at the origin and moves at time $n \geqslant 1$ by a unit step $\mathbf{e}_n$ to a neighboring lattice point; for example, if $d=3$, the possible steps are

$$\mathbf{e} = (\pm 1,0,0), \quad (0,\pm 1,0), \quad \text{and} \quad (0,0,\pm 1).$$

The position of the particle at time $n \geqslant 1$ is the sum of the individual steps: $\mathbf{s}_n = \mathbf{e}_1 + \cdots + \mathbf{e}_n$. The step $\mathbf{e}_n$ is statistically independent of the preceding steps $\mathbf{e}_j : j < n$, and the possible steps are equally likely at each stage. This means that

$$\begin{aligned} P(\mathbf{e}_1 = e_1, \ldots, \mathbf{e}_n = e_n) &= P(\mathbf{e}_1 = e_1) \times \cdots \times P(\mathbf{e}_n = e_n) \\ &= (2d)^{-n} \end{aligned}$$

for any fixed unit steps $e_1, \ldots, e_n$, in which $P(E)$ means "the probability of the event E." The problem is to compute $P(\mathbf{s}_n = k)$ and to study the behavior of $\mathbf{s}_n$ for $n\uparrow\infty$. Pólya's idea is to think of $P(\mathbf{s}_n = k)$ as the Fourier coefficient $\hat{f}(k)$ of a function $f \in \mathrm{L}^2(T^d)$:

$$f(x) = \sum_{k\in Z^d} P(\mathbf{s}_n = k)\, e^{2\pi i k\cdot x}.$$

This sum is just the "expectation" or "mean value" of $\exp(2\pi i \mathbf{s}_n \cdot x)$ and is easily computed using the independence of the individual steps:

$$\begin{aligned} f(x) &= \sum_{e_1} \cdots \sum_{e_n} (2d)^{-n} \exp(2\pi i e_1 \cdot x) \cdots \exp(2\pi i e_n \cdot x) \\ &= \Big[(2d)^{-1} \sum_{e_1} \exp(2\pi i e_1 \cdot x)\Big]^n \\ &= [(\cos 2\pi x_1 + \cdots + \cos 2\pi x_d)/d]^n \\ &= [f_d(x)]^n. \end{aligned}$$

Pólya's formula is immediate from this;

$$P(\mathbf{s}_n = k) = \hat{f}(k) = (f_d^n)^\wedge(k) = \int_{T^d} f_d^n e^{-2\pi i k\cdot x}.$$

In particular,

$$P(\mathbf{s}_n = 0) = \int_{T^d} f_d^n,$$

and since $|f_d| \leqslant 1$, the expected number of times the particle visits the origin can be expressed as

$$\begin{aligned}\sum_{n=0}^{\infty} P(\mathbf{s}_n = 0) &= \lim_{\varepsilon\uparrow 1} \sum_{n=0}^{\infty} \varepsilon^n P(\mathbf{s}_n = 0)\\ &= \lim_{\varepsilon\uparrow 1} \int_{T^d} \sum_{n=0}^{\infty} \varepsilon^n f_d^n\\ &= \lim_{\varepsilon\uparrow 1} \int_{T^d} (1-\varepsilon f_d)^{-1}\\ &= \int_{T^d} (1-f_d)^{-1}\end{aligned}$$

by an application of monotone convergence to the region where $0 \leqslant f_d \leqslant 1$. Because

$$\frac{1}{2} \times \frac{4\pi^2|x|^2}{2d} \leqslant 1 - f_d \leqslant \frac{4\pi^2|x|^2}{2d}$$

for small $|x|$, the integral diverges for $d \leqslant 2$ and converges for $d \geqslant 3$. Pólya used this to prove a very striking fact about the ultimate behavior of the walk:

$$\begin{aligned} P(\mathbf{s}_n = 0, \text{i.o.}) &= 1 \qquad \text{for } d \leqslant 2,\\ P(\lim_{n\uparrow\infty} |\mathbf{s}_n| = \infty) &= 1 \qquad \text{for } d \geqslant 3. \end{aligned}$$

PROOF FOR $d \geqslant 3$. The integral $\int (1-f_d)^{-1} < \infty$ says that the expected number of times that the particle visits the origin is less than ∞. This can happen only if the actual number of visits is less than ∞ with probability 1, and since the origin is not special in any way, the same must be true of every lattice point in Z^d. But this means that for any $R < \infty$, the particle ultimately stops visiting the ball $|k| < R$, and that is the same as to say

$$P(\lim_{n\uparrow\infty} |\mathbf{s}_n| = \infty) = 1.$$

PROOF FOR $d \leqslant 2$. At time $n = 1$, the particle steps to one of the $2d$ nearest neighbors of the origin. The problem is to check that the probability p of ultimately returning to the origin is 1. But that is self-evident as soon as you reflect that the probability of visiting the origin m or more times (including the visit at time $n = 0$) is p^{m-1}, for then the probability of precisely m visits is

$$p^{m-1} - p^m = p^{m-1}(1-p),$$

and if p were less than 1, the expected number of visits would be

$$\sum_{m=1}^{\infty} mp^{m-1}(1-p) = (1-p)^{-1} < \infty,$$

contradicting the evaluation $\int (1-f_d)^{-1} = \infty$ ($d \leqslant 2$). The proof is finished; for additional information on the subject, see Feller [1968, Vol. 1, pp. 342–371].

3*. Fourier Series on a Nonstandard Torus

A variant of the several-dimensional Fourier series of Subsection 1 arises by looking at functions on a nonstandard torus. For simplicity, only dimension $n = 2$ is discussed. Pick numbers $-\infty < n < \infty$ and $b > 0$ and introduce the [nonstandard] lattice $Z \subset R^2$ of all plane points of the form

$$\omega = j(1,0) + k(a,b) \qquad \text{with integral } j \text{ and } k,$$

as in Fig. 1. As for the standard lattice of article 1, a function with "periods from Z" can be thought of as living on the torus $T = R^2/Z$ obtained by identifying opposite sides of the "fundamental cell" shaded in the figure. Define Z' to be the "dual lattice" of points $\omega' \in R^2$ such that the inner product $\omega' \cdot \omega$ is integral for every $\omega \in Z$.

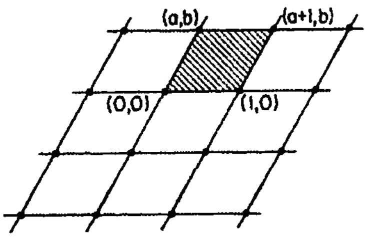

FIGURE 1

EXERCISE 3. Z' is the lattice of points

$$\omega' = j(1, -a/b) + k(0, 1/b) \qquad \text{with integral } j \text{ and } k;$$

in particular, the standard lattice ($a = 0$, $b = 1$) is its own dual. Check this.