TMA4170: Corona Week 1

Limit theorems and L^p spaces

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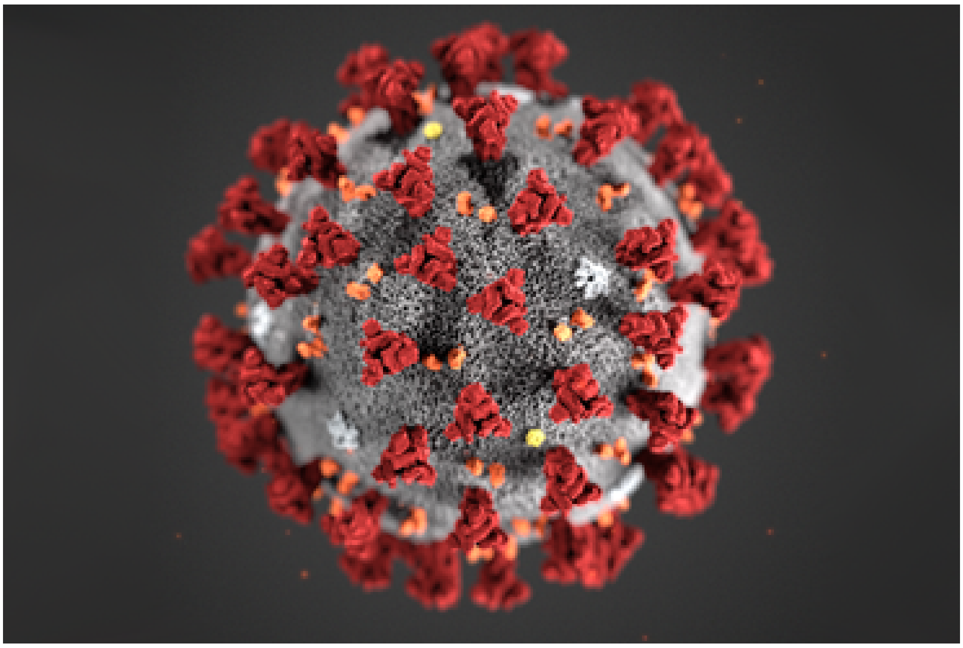

1. Limit theorems

One of the main advantages of the Lebesgue integral over the Riemann integral is the availability of limit theorems, which allow us to calculate or estimate integrals of possibly complicated functions with mimimal effort.

THEOREM 1.1 (Monotone convergence theorem). *If f_n is a sequence of non-negative functions and $f_1 \leqslant f_2 \leqslant \dots$ almost everywhere then the limit function $f = \lim_n f_n$ satisfies $\int_X f\,d\mu = \lim_n \int_X f_n\,d\mu$.*

The proof is simply an adaptation of the construction of the integral of a non-negative measurable function by approximating it with simple functions.

Notice that we do not assume functions f_n to be integrable. The limit function f is well defined almost everywhere if we allow it to attain infinite values.

THEOREM 1.2 (Fatou Lemma). *If f_n is a sequence of non-negative functions then*

$$\int_X \liminf_n f_n\,d\mu \leqslant \liminf_n \int_X f_n\,d\mu.$$

PROOF. By denoting

$$g_n = \inf_{k \geqslant n} f_k, \quad f = \liminf_n f_n,$$

we obtain $g_n \leqslant f_n$, $0 \leqslant g_1 \leqslant g_2 \leqslant \dots$ and $\lim_n g_n = f$. Hence from the monotone convergence theorem we get

$$\int_X f_n\,d\mu \geqslant \int_X g_n\,d\mu \to \int_X f\,d\mu,$$

and then the result follows immediately. □

EXAMPLE 1.3. Suppose $f_n = \mathbb{1}_{[n,n+1]}$. Then $\liminf_n f_n = 0$, while $\int_{\mathbb{R}} f_n\,d\lambda = 1$ for every n. This simple example shows that the Fatou lemma indeed requires an inequality. It is also an easy way to remember, in which direction the inequality is pointing.

THEOREM 1.4 (Lebesgue dominated convergence theorem). *Let f_n and g be measurable functions such that $\int_X g\,d\mu < \infty$ and for every n the inequality $|f_n| \leqslant g$ is satisfied almost everywhere. If $f = \lim_n f_n$ almost everywhere then*

$$\lim_n \int_X |f_n - f|\,d\mu = 0 \quad \textit{and} \quad \int_X f\,d\mu = \lim_n \int_X f_n\,d\mu.$$

PROOF. Let $h_n = |f_n - f|$ and $h = 2g$. Then $h_n \to 0$ almost everywhere and $0 \leqslant h_n \leqslant h$. Thus by applying the Fatou lemma to the sequence $h - h_n$, we obtain

$$\int_X h\,d\mu = \int_X \liminf_n (h - h_n)\,d\mu \leqslant \liminf_n \int_X (h - h_n)\,d\mu$$
$$= \int_X h\,d\mu - \limsup_n \int_X h_n\,d\mu.$$

This gives us $\limsup_n \int_X h_n\,d\mu = 0$, because $\int_X h\,d\mu < \infty$. But becasue h_n are non-negative we also have $\liminf_n \int_X h_n\,d\mu = 0$ and so $\lim_n \int_X h_n\,d\mu = 0$.

Thus we have shown that $\int_X |f_n - f|\,d\mu \to 0$. Because

$$\int_X f_n\,d\mu - \int_X f\,d\mu \leqslant \int_X |f_n - f|\,d\mu,$$

the second relation follows from the first. □

EXAMPLE 1.5. Let $X = [0,1]$ and $f_n = n\mathbb{1}_{[0,1/n]}$. Then we have $f_n \to 0$ λ-almost everywhere, but $\int_{[0,1]} f_n\,d\lambda = 1$. The assumption of "dominated convergence", appearing in (the very name of) Theorem 1.4 is therefore important.

COROLLARY 1.6. *Let $\mu(X) < \infty$ and let functions $|f_n| \leqslant M$ for some number $M \geqslant 0$. If $f = \lim_n f_n$ almost everywhere then $\int_X f\,d\mu = \lim_n \int_X f_n\,d\mu$.*

The following theorem tells us that we can use the integral to produce new measures. When reading the statement below think for example of the bell-curve $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ and $\mu = \lambda$. In this way we may define the normal probability distribution $\mathcal{N}(0,1)$ as a measure.

THEOREM 1.7. *If f is a measurable and non-negative function on a measure space (X, Σ, μ) then the set function $\nu : \Sigma \to [0,\infty]$ given for every $A \in \Sigma$ by $\nu(A) = \int_A f\,d\mu$ is a measure on Σ.*

PROOF. By the properties of the integral, we know that ν is an additive set function on Σ. If $A_n \uparrow A$ for some sets $A_n, A \in \Sigma$ then $\mathbb{1}_{A_n}$ is a non-decreasing sequence of functions converging to $\mathbb{1}_A$, while $f\mathbb{1}_{A_n} \to f\mathbb{1}_A$. By the monotone convergence theorem we thus have

$$\nu(A) = \int_A f\,d\mu = \int_X f\mathbb{1}_A\,d\mu = \lim_n \int_X f\mathbb{1}_{A_n}\,d\mu = \lim_n \nu(A_n)$$

Hence ν is continuous from below and thus countably additive (it is a measure). □

2. Convergence in measure

In this section we consider another notion of convergence for sequences of measurable functions. For us, it will only be important in the proof of completeness of L^p spaces at the end of this part of the notes. If you are willing to accept this result without a rigorous proof, you may skip this section.

DEFINITION 2.1. We say that a sequence of measurable functions $f_n : X \to \mathbb{R}$ converges in measure to a function f if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu\big(\{x : |f_n(x) - f(x)| \geqslant \varepsilon\}\big) = 0.$$

In such case we denote $f_n \xrightarrow{\mu} f$.

PROPOSITION 2.2. *A sequence which converges almost uniformly, converges in measure.*

PROOF. If functions f_n converge to f almost uniformly, then for every $\varepsilon > 0$ there exists a set A such that $\mu(A) < \varepsilon$ and $|f_n(x) - f(x)| < \varepsilon$ for large enough n and all $x \notin A$. Thus $\{x : |f_n(x) - f(x)| \geqslant \varepsilon\} \subseteq A$ and $\mu\big(\{x : |f_n(x) - f(x)| \geqslant \varepsilon\}\big) \leqslant \mu(A) < \varepsilon$. $\square$

REMARK 2.3. *Let $f_n : [0, 1] \to \mathbb{R}$ denote the sequence*

$$\mathbb{1}_{[0,1]},\ \mathbb{1}_{[0,1/2]},\ \mathbb{1}_{[1/2,1]},\ \mathbb{1}_{[0,1/4]},\ \mathbb{1}_{[1/4,1/2]}, \ldots$$

We can check that f_n converges to 0 in Lebesgue measure, but

$$\liminf_n f_n(x) = 0, \quad \limsup_n f_n(x) = 1 \quad \text{for every } x \in [0, 1],$$

so the sequence doesn't converge almost uniformly.

LEMMA 2.4 (Chebyshev inequality). *If f is a measurable function then for every $\varepsilon > 0$*

$$\varepsilon \cdot \mu\big(\{x : |f(x)| \geqslant \varepsilon\}\big) \leqslant \int_X |f|\, d\mu.$$

PROOF. Let $A_\varepsilon = \{x : |f(x)| \geqslant \varepsilon\}$. Then $|f|\,\mathbb{1}_{A_\varepsilon} \geqslant \varepsilon\,\mathbb{1}_{A_\varepsilon}$ and $\int_X |f|\, d\mu \geqslant \int_{A_\varepsilon} |f|\, d\mu \geqslant \varepsilon\mu(A_\varepsilon)$ $\square$

THEOREM 2.5 (Riesz). *Let (X, Σ, μ) be a finite measure space and let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions satisfying the Cauchy condition in measure, i.e.*

$$\lim_{n,m\to\infty} \mu\big(\{x : |f_n(x) - f_m(x)| \geqslant \varepsilon\}\big) = 0$$

for every $\varepsilon > 0$. Then

- *there exists a subsequence $n(k) \in \mathbb{N}$, such that the sequence of functions $f_{n(k)}$ is convergent almost everywhere;*
- *the sequence f_n converges in measure to some function f.*

PROOF. Notice that the Cauchy condition we assumed implies that for every k there exists $n(k)$, such that for any $n, m \geqslant n(k)$ we have

$$\mu\big(\{x : |f_n(x) - f_m(x)| \geqslant 1/2^k\}\big) \leqslant 1/2^k,$$

and in addition we can take $n(1) < n(2) < \dots$. Let

$$E_k = \{x : |f_{n(k)}(x) - f_{n(k+1)}(x)| \geqslant 1/2^k\}, \quad A_k = \bigcup_{n \geqslant k} E_n.$$

Then $\mu(A_k) \leqslant 1/2^{k-1}$ and hence the set $A = \bigcap_k A_k$ has measure zero. If $x \notin A$ then for every k such that $x \notin A_k$ and every $i \geqslant k$ we have

$$|f_{n(i)} - f_{n(i+1)}| \leqslant 1/2^i.$$

It follows from the triangle inequality that for $j > i \geqslant k$ we have

$$|f_{n(i)} - f_{n(j)}| \leqslant 1/2^{i-1}.$$

This means that for $x \notin A$ the numerical sequence $f_{n(i)}(x)$ satisfies the Cauchy condition and hence converges to a number, which we (unsurprisingly) denote as $f(x)$. In this way we obtain that $f_{n(k)}$ converges almost everywhere to the fuction f and this proves the first part of the theorem.

In order to verify the second part it suffices to notice that $f_n \xrightarrow{\mu} f$, which follows from

$$\{x : |f_n(x) - f(x)| \geqslant \varepsilon\}$$
$$\subseteq \{x : |f_n(x) - f_{n(k)}(x)| \geqslant \frac{\varepsilon}{2}\} \cup \{x : |f_{n(k)}(x) - f(x)| \geqslant \frac{\varepsilon}{2}\},$$

and the Cauchy condition for the convergence in measure. □

3. The p-norm and useful inequalities

LEMMA 3.1 (Young inequality for products). *For any positive numbers a, b, p, q, if $1/p + 1/q = 1$ then*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

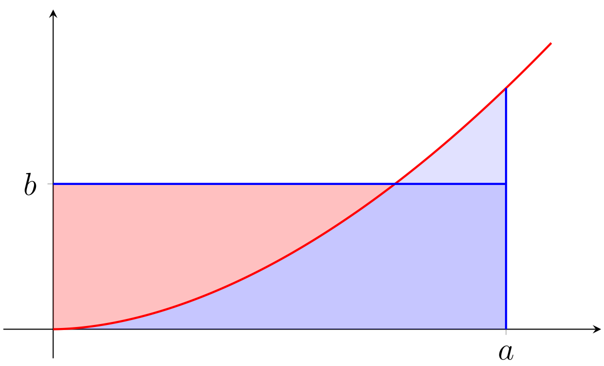

FIGURE 1. Young inequality: the rectangle $[0, a] \times [0, b]$ is covered by the blue and red areas, but there is an excess of blue, hence the inequality.

Proof. Consider the function $f(t) = t^{p-1}$ on the interval $[0, a]$. We assume $p > 1$ therefore f has the inverse function $g(s) = s^{1/(p-1)}$. Note that the areas under the graphs of $f : [0, a] \to \mathbb{R}$ and $g : [0, b] \to \mathbb{R}$ cover the rectangle $[0, a] \times [0, b]$ (see Figure 1).

Thus

$$ab \leqslant \int_0^a t^{p-1}\, dt + \int_0^b s^{1/(p-1)}\, ds = \frac{t^p}{p}\bigg|_0^a + \frac{s^q}{q}\bigg|_0^b = \frac{a^p}{p} + \frac{b^q}{q},$$

because $1 + 1/(p-1) = p/(p-1) = q$. □

Definition 3.2. For every measurable function (integrable or not) $f : X \to \mathbb{R}$ and $p \geqslant 1$ the expression

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

is called the p-th integral norm, or p-norm for short, of the function f.

Theorem 3.3 (Hölder inequality). *For every pair of functions f, g and numbers $p, q > 0$ such that $1/p + 1/q = 1$ we have the following inequality*

$$\|fg\|_1 = \int_X |f \cdot g| \, d\mu \leqslant \|f\|_p \cdot \|g\|_q.$$

Proof. The inequality is obviously true if one of the norms on the right-hand side is infinite. Otherwise, for a given $x \in X$ we substitute

$$a = \frac{|f(x)|}{\|f\|_p}, \quad b = \frac{|g(x)|}{\|g\|_q}$$

into the inequality in the previous lemma in order to obtain (for every $x \in X$)

$$\frac{|f(x) \cdot g(x)|}{\|f\|_p \cdot \|g\|_q} \leqslant \frac{1}{p} \cdot \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \cdot \frac{|g(x)|^q}{\|g\|_q^q}.$$

By integrating the last inequality we get

$$\int_X |fg| \, d\mu \|f\|_p \cdot \|g\| q \leqslant 1p + 1q = 1.$$ □

Theorem 3.4 (Minkowski inequality). *For every pair of functions f, g and a number $p \geqslant 1$, we have the following inequality*

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p.$$

Proof. The inequality is satisfied for $p = 1$. For $p > 1$ we may find a number q satisfying the condition $1/p + 1/q = 1$. Notice that

$(p-1)q = p$ and $p/q = p-1$. We use the Hölder inequality to get

$$\begin{aligned}
&\|f+g\|_p^p = \int_X |f+g|^p \, d\mu \\
&\leqslant \int_X |f| \cdot |f+g|^{p-1} \, d\mu + \int_X |g| \cdot |f+g|^{p-1} \, d\mu \\
&\leqslant \|f\|_p \bigg(\int_X |f+g|^{(p-1)q} \, d\mu \bigg)^{\frac{1}{q}} + \|g\|_p \bigg(\int_X |f+g|^{(p-1)q} \, d\mu \bigg)^{\frac{1}{q}} \\
&= \big(\|f\|_p + \|g\|_p\big) \cdot \bigg(\int_X |f+g|^p \, d\mu \bigg)^{\frac{1}{q}} = \big(\|f\|_p + \|g\|_p\big) \cdot \|f+g\|_p^{p/q}.
\end{aligned}$$

We now divide both sides by $\|f+g\|_p^{p/q}$ and we get result.

Note that in order for this proof to be entirely correct, we need to verify that $\|f\|_p, \|g\|_p < \infty$ implies $\|f+g\|_p < \infty$. $\square$

So far we defined the integral for real-valued functions, but in the context of Fourier analysis, we have to deal with complex values. Here is a technical description of what needs to be done. The take-home message is: the integral is defined in the most natural way and everything works as expected.

Consider measure spaces (X, Σ, μ) and (Y, Θ, ν). We may then define

$$\Sigma \otimes \Theta = \sigma\big(\{A \times B : \; A \in \Sigma, \; B \in \Theta\}\big),$$

which is a σ-field of subsets of $X \times Y$. Similarly, we may define

$$(\mu \otimes \nu)(A \times B) = \mu(A) \cdot \nu(B),$$

and show that $\mu \otimes \nu$ extends to a measure on $(X \times Y, \Sigma \otimes \Theta)$. It can also be shown that

$$\mathrm{Bor}(\mathbb{R} \times \mathbb{R}) = \mathrm{Bor}(\mathbb{R}) \otimes \mathrm{Bor}(\mathbb{R}).$$

This allows us to easily consider spaces of functions of complex values. For a measure space (X, Σ, μ) and a function $f : X \to \mathbb{C}$ we say that f is measurable if $f^{-1}[B] \in \Sigma$ for every Borel set $B \subseteq \mathbb{C}$. Here $\mathbb{C}$ may be identified with $\mathbb{R} \times \mathbb{R}$ and so $\mathrm{Bor}(\mathbb{C}) = \mathrm{Bor}(\mathbb{R}) \otimes \mathrm{Bor}(\mathbb{R})$.

We may express such a function as $f = \operatorname{Re} f + i \operatorname{Im} f$, where $\operatorname{Re} f$ and $\operatorname{Im} f$ are real-valued functions. Then f is measurable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable.

Hence if f is measurable then its modulus $|f| = \sqrt{(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2}$ is measurable as well. The function f is integrable when $\int_X |f| \, d\mu < \infty$, while

$$\int_X f \, d\mu = \int_X \operatorname{Re} f \, d\mu + i \int_X \operatorname{Im} f \, d\mu$$

defines the integral. The basic properties of the integral remain valid.

Notice that the p-norms of complex-valued functions may be considered with the definition unchanged.

4. Banach spaces of p-integrable functions

Recall that a norm on a linear space X is a function $\|\cdot\| : X \to \mathbb{C}$ (or $X \to \mathbb{R}$) such that

(1) $\|x\| \geqslant 0$ for every $x \in \mathbb{R}^d$ and $\|x\| = 0$ if and only if $x = 0$;
(2) (triangle inequality) $\|x + y\| \leqslant \|x\| + \|y\|$ for every $x, y \in X$;
(3) (homogeneity) $\|ax\| = |a|\|x\|$ for every $x \in X$ and every $a \in \mathbb{C}$ (or $a \in \mathbb{R}$).

Definition 4.1. A normed space $(X, \|\cdot\|)$ is called a Banach space if the metric induced by the norm is complete, i.e. for every sequence $x_n \in X$ satisfying the Cauchy condition

$$\lim_{n,k\to\infty} \|x_n - x_k\| = 0,$$

there exists $x \in X$ such that $\|x_n - x\| \to 0$ (x is the limit of the sequece).

The p-norm function $\|\cdot\|_p$ is in fact a norm: Minkowski inequality is the triangle inequality for $\|\cdot\|_p$ and homogeneity follows directly from the properties of the integral.

The only problem is with the first axiom, since $\|f\|_p = 0$ is only equivalent to saying that $f = 0$ almost everywhere.

Definition 4.2. For a given measure space (X, Σ, μ), by $L^p(\mu)$ we denote the space of all measurable functions $f : X \to \mathbb{R}$ for which $\|f\|_p < \infty$. Elements of $L^p(\mu)$ which are equal almost everywhere are identified as classes of abstraction.

In this way $L^p(\mu)$ equipped with the p-th integral norm becomes a normed space (strictly speaking, we first have to show it is a linear space, see Problem 1), but formally speaking it consists not of functions, but classes of abstaction (of functions). Most often. however, we may still refer to the elements of $L^p(\mu)$ as functions without any confusion.

It is nonetheless important not to forget about this distinction. For example, if f is a measurable function and $[f]$ is its class of abstraction such that $f \in [f] \in L^p(\lambda)$, then for a chosen point $x \in \mathbb{R}$ the value $[f](x)$ is *undefined*, since a single point has Lebesgue measure zero. In fact, $[f]$ contains functions attaining all possible values at x.

Notice that if $f_n \to f$ almost everywhere, then the same is true for every representative of the respective classes of abstraction, while it is not true for the actual pointwise convergence (everywhere without "almost").

In different contexts, $L^p(\mu)$ may also be denoted by $L^p(X, \Sigma, \mu)$ or as $L^p(X)$. For example, we usually write $L^p(\mathbb{R})$ or $L^p(\mathbb{T})$ to refer to spaces defined using the Lebesgue measure on $\mathbb{R}$ or $\mathbb{T}$.

THEOREM 4.3. *For every $p \geqslant 1$ the space $L^p(\mu)$ is a Banach space.*

PROOF. Consider $p = 1$ and let $f_n \in L^1(\mu)$ be a Cauchy sequence in the norm $\|\cdot\|_1$, that is

$$\lim_{n,k\to\infty} \int_X |f_n - f_k| \, d\mu = 0.$$

Then for $\varepsilon > 0$ it follows from the Chebyshev inequality that

$$\int_X |f_n - f_k| \, d\mu \geqslant \varepsilon \cdot \mu\Big(\big\{x : |f_n(x) - f_k(x)| \geqslant \varepsilon\big\}\Big),$$

which means that f_n is a Cauchy sequence in measure.

It follows from the Riesz theorem that there exists an increasing sequence $n_k \in \mathbb{N}$ and a function f such that $f_{n_k} \to f$ almost everywhere. On the other hand, the Fatou lemma gives us

$$\int_X |f| \, d\mu \leqslant \liminf_k \int_X |f_{n_k}| \, d\mu < \infty,$$

because the Cauchy condition implies that the sequence of integrals $\int_X |f_n| \, d\mu$ is bounded.

Using the Fatou lemma once again we obtain

$$\int_X |f - f_{n_k}| \, d\mu = \int_X \liminf_j |f_{n_j} - f_{n_k}| \, d\mu$$
$$\leqslant \liminf_j \int_X |f_{n_j} - f_{n_k}| \, d\mu \leqslant \varepsilon,$$

for k large enough. Finally, because

$$\int_X |f - f_n| \, d\mu \leqslant \int_X |f - f_{n_k}| \, d\mu + \int_X |f_{n_k} - f_n| \, d\mu,$$

we obtain that f is in fact the limit of the sequence f_n in the space $L^1(\mu)$. The proof for $p > 1$ is a rather mechanical modification of this argument. □

Finally, we introduce the $L^\infty(\mu)$ space.

DEFINITION 4.4. We define the p-norm for $p = \infty$ by

$$\|f\|_\infty = \inf\big\{C \geqslant 0 : \ |f(x)| \leqslant C \text{ for almost every } x \in \mathbb{R}^d\big\}.$$

We define the space $L^\infty(\mu)$ by

$$L^\infty(\mu) = \{f : X \to \mathbb{R} : \ f \text{ is measurable and } \|f\|_\infty < \infty\},$$

where again we identify functions which are equal almost everywhere.

Despite a somewhat different definition of $L^\infty(\mu)$, we may prove the following theorems which show that it belongs with other $L^p(\mu)$ spaces.

THEOREM 4.5. *$L^\infty(\mu)$ is a Banach space.*

THEOREM 4.6. *Let $p_0 \geqslant 1$ and $f \in L^p(\mu)$ for all $p_0 \leqslant p < \infty$. Then $f \in L^\infty(\mu)$ and $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.*

Questions:

- Can you give an example of a sequence of functions, which illustrates that $f_n \geqslant 0$ is a necessary assumption in the Fatou lemma?
- Can you write a "reverse" Fatou lemma, where $\limsup$ is involved?
- Can you use the dominated convergence theorem to strenghten the Fatou lemma by replacing the assumption $f_n \geqslant 0$ with $f_n \geqslant g$ and g is integrable?
- Can you *draw* the counterexample to the dominated convergence theorem described in Example 1.5?
- Can you prove Corollary 1.6?
- Can you *draw* the Chebyshev inequality on a picture?
- Can you prove that $\mathbb{R}^d$ and $\mathbb{C}^d$ are Banach spaces?
- Can you think of any other spaces that are Banach spaces?

Problems:

Problem 1. Check that $|a+b|^p \leqslant 2^{p/q}(|a|^p + |b|^p)$ for $1/p+1/q = 1$; deduce that $L^p(\mu)$ is a linear space.

Problem 2. Show that simple functions are a dense subset of $L^p(\mu)$; show that $C[0,1]$ is a dense subset of $L^p[0,1]$.

Problem 3. Show that $C(\mathbb{R})$ is not a subset of any $L^p(\mathbb{R})$; show that $C_c^\infty(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ are dense in $L^p(\mathbb{R})$.

Problem 4. Let (X, Σ, μ) be a finite measure space. Prove that $L^p(X) \subseteq L^q(X)$ for $p \leqslant q$.