TMA4170: Corona Week 2

Fourier transform in L^p spaces.

Miłosz Krupski

Based on
L. Grafakos *Classical Fourier Analysis, 3rd Edition*, Springer 2014,
J. Duoandikoetxea *Fourier Analysis*, AMS 2001

1. Some observations on $L^p(\mu)$-spaces

1.1. Density. Let us consider the case of $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}), \lambda)$. Above all notice that by the very definition of the integral, simple functions are a dense subset of all $L^p(\mathbb{R})$-spaces for $1 \leqslant p < \infty$.

Then, observe that compactly supported continuous functions are p-integrable for every $p \geqslant 1$ and each such function is bounded. Hence, $C_c(\mathbb{R}) \subseteq L^p(\mathbb{R})$ and because we may approximate (pointwise) every indicator function of an interval $\mathbb{1}_{[a,b]}$ by continuous, compactly supported functions, we can also say that $C_c(\mathbb{R})$ is a dense subset of $L^p(\mathbb{R})$.

In a similar way we may conclude the same about every space $C_c^k(\mathbb{R})$ of k-differentiable, compactly supported functions as well as the space $C_c^\infty(\mathbb{R})$ of smooth compactly supported functions.

Finally, however it requires some more calculations, we notice that the Schwartz class $\mathcal{S}(\mathbb{R})$ is also a dense subset of every $L^p(\mathbb{R})$-space (density itself follows from the fact that $C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, but we need to show that every function in $\mathcal{S}(\mathbb{R})$ is p-integrable in the first place).

1.2. Duality. If E is a Banach space, then we may consider the space E^* of all linear functionals on E, i.e. linear operators with numeral values ($\mathbb{R}$ or $\mathbb{C}$), or precisely

$$E^* = \{\phi : E \to \mathbb{C} : \ \phi \text{ is a linear mapping}\}.$$

In such case, E^* is also a Banach space and we say that it is the **dual space** of E. The norm on the space E^* may be defined by the following identity

$$\|\phi\|_{E^*} = \sup\Big\{\phi(x) : x \in E, \ \|x\|_E \leqslant 1\Big\}.$$

Let $1 < p, q < \infty$ be such that $1 = \frac{1}{p} + \frac{1}{q}$. Then it turns out that the spaces $L^p(\mu)$ and $L^q(\mu)$ are the duals of each other, i.e. $L^p(\mu)^* = L^q(\mu)$ and vice-versa.

The case when $p = 1$ and $q = \infty$ or $p = \infty$ and $q = 1$ is a bit more complicated, where we do have $L^1(\mu)^* = L^\infty(\mu)$, but $L^\infty(\mu)^* \neq L^1(\mu)$.

We are going to prove only half of this result, to say that every $g \in L^q(\mu)$ defines in a natural way a linear functional on $L^p(\mu)$. This means that $g \in L^p(\mu)^*$ and so $L^q(\mu) \subseteq L^p(\mu)^*$. The other half would say that *every* linear functional in $L^p(\mathbb{R}^d)^*$ has such a representation and hence $L^p(\mu)^* \subseteq L^q(\mu)$.

THEOREM 1.1. *Let $1 < p < \infty$ and let $g \in L^q(\mu)$. Then the mapping*

$$G(f) = \int_X f\, g\, d\mu$$

is a bounded linear functional and $\|G\| = \|g\|_q$.

Proof. We can assume $g \neq 0$. It follows from the Hölder inequality that

$$\int_X |f\,g|\,d\mu \leqslant \left(\int_X |f|^p\,d\mu\right)^{1/p}\left(\int_X |g|^q\,d\mu\right)^{1/q} < \infty,$$

therefore the value $G(f)$ is well-defined. G is a linear operator because of the linearity of the integral. Moreover,

$$|G(f)| \leqslant \int_X |f\,g|\,d\mu \leqslant \|g\|_q\|f\|_p,$$

therefore $\|G\| \leqslant \|g\|_q$.

Let $f(x) = \overline{\operatorname{sgn} g(x)}|g(x)|^{q-1}$. Then

$$\int_X |f|^p\,d\mu = \int_X |g|^{p(q-1)}\,d\mu = \int_X |g|^q\,d\mu < \infty.$$

Hence $f \in L^p(\mu)$ and $\|f\|_p^p = \|g\|_q^q$. Moreover,

$$G(f) = \int_X |g|^q\,d\mu = \|g\|_q^q.$$

Finally,

$$\frac{G(f)}{\|f\|_p} = \frac{\|g\|_q^q}{\|g\|_q^{q/p}} = \|g\|_q,$$

thus $\|G\| \geqslant \|g\|_q$. □

With this result in mind (and the missing half), we conclude that we have another way of calculating the p-norm

$$\|f\|_p = \sup\left\{\left|\int_X fg\,d\mu\right| : g \in L^q(\mu),\ \|g\|_q \leqslant 1\right\}.$$

Let us conclude this section with noting that the spaces $L^p(\mu)$ are generally not comparable. We have $L^p(\mu) \subseteq L^q(\mu)$ for $p \geqslant q$, only if μ is a finite measure, for example the Lebesgue measure on the torus $\mathbb{T}$.

Otherwise, this is not the case and in particular we have

$$L^1(\mathbb{R})\backslash L^p(\mathbb{R}) \neq \varnothing \quad \text{and} \quad L^p(\mathbb{R})\backslash L^1(\mathbb{R}) \neq \varnothing$$

for every $p > 1$.

2. Fourier transform in $L^1(\mathbb{R})$

Before switching to discuss elements of measure and integration theory, we defined the Fourier transform for functions in the Schwartz class $\mathcal{S}(\mathbb{R})$. Precisely, for $f \in \mathcal{S}(\mathbb{R})$ we denote

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\,dx$$

and call $\widehat{f}$ the Fourier transform of the function f.

Notice that the same definition remains valid for $f \in L^1(\mathbb{R})$. Indeed, for every $\xi \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} |e^{-2\pi i x\xi} f(x)|\, dx = \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1,$$

which means that

$$x \mapsto e^{2\pi i x\xi} f(x)$$

is an integrable function for every $\xi \in \mathbb{R}$ and hence the relevant integral – i.e. the Fourier transform $\widehat{f}(\xi)$ – is well-defined. Moreover, we have

$$|\widehat{f}(\xi)| = \left| \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\, dx \right| \leqslant \int_{\mathbb{R}} |e^{-2\pi i x\xi} f(x)|\, dx = \|f\|_1,$$

which holds for every $\xi \in \mathbb{R}$, and hence according to the definition of $\|\cdot\|_\infty$ we have $\|\widehat{f}\|_\infty \leqslant \|f\|_1$.

Basic properties of the Fourier transform in $L^1(\mathbb{R})$ are the same as in the Schwartz class.

PROPOSITION 2.1. *Let $f, g \in L^1(\mathbb{R})$, $a \in \mathbb{C}$, $y \in \mathbb{R}$, $n \in \mathbb{N}$ and $t > 0$. Denote $\tau_y f(x) = f(x+y)$ and $\widetilde{f}(x) = f(-x)$. We have*

1. $\widehat{f+g} = \widehat{f} + \widehat{g}$;
2. $\widehat{af} = a\widehat{f}$;
3. $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$;
4. $\widehat{\overline{f}} = \overline{\widetilde{\widehat{f}}}$;
5. $\widehat{\tau_y f}(\xi) = e^{-2\pi i y\xi}\widehat{f}(\xi)$;
6. $f * g \in L^1(\mathbb{R})$ *and* $\widehat{f * g} = \widehat{f}\,\widehat{g}$

We may also prove additional results.

PROPOSITION 2.2. *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}$.*

PROOF. Let $\varepsilon > 0$. For $|\xi - \zeta| < \varepsilon$ we have

$$\begin{aligned} |\widehat{f}(\xi) - \widehat{f}(\zeta)| &= \left| \int_{\mathbb{R}} \left(e^{-2\pi i x\xi} - e^{-2\pi i x\zeta} \right) f(x)\, dx \right| \\ &\leqslant \int_{\mathbb{R}} \left| 1 - e^{-2\pi i x(\zeta - \xi)} \right| |f(x)|\, dx \leqslant |\xi - \zeta| \|f\|_1 < \varepsilon \|f\|_1. \quad \square \end{aligned}$$

In the context of Fourier series we proved a result called the Riemann-Lebesgue lemma, which says that the Fourier coefficients $\widehat{f}(n)$ of an integrable function $f \in L^1(\mathbb{T})$ (or $L^2(\mathbb{T})$, or $C(\mathbb{T})$) converge to 0 as $|n| \to \infty$. A similar result is available for the Fourier transform.

THEOREM 2.3 (Riemann-Lebesgue lemma). *If $f \in L^1(\mathbb{R})$ then*

$$|\widehat{f}(\xi)| \to 0 \quad \textit{as} \quad |\xi| \to \infty.$$

PROOF. Consider the function $\mathbb{1}_{[a,b]}$ on $\mathbb{R}$. We then have

$$\widehat{\mathbb{1}_{[a,b]}}(\xi) = \frac{e^{-2\pi i\xi a} - e^{-2\pi i\xi b}}{2\pi i\xi},$$

which tends to zero as $|\xi| \to \infty$. As a consequence, the result is true for any simple function.

For a general function $f \in L^1(\mathbb{R})$ we consider an approximating sequence of simple functions s_n and notice that

$$|\widehat{f}(\xi)| \leqslant |\widehat{f}(\xi) - \widehat{s}_n(\xi)| + |\widehat{s}_n(\xi)| \leqslant \|f - s_n\|_1 + |\widehat{s}_n(\xi)|. \qquad \square$$

3. Inverse Fourier transform in $L^1(\mathbb{R})$

In the same way as the Fourier transform, we may define the inverse Fourier transform $\check{g}$ of a function $g \in L^1(\mathbb{R})$. However – we have to be very careful not to go too far.

It is not necessarily possible to define the inverse Fourier transform $(\widehat{f})^{\vee}$ of the function $\widehat{f}$ itself being the Fourier transform of a function f in $L^1(\mathbb{R})$. The formula

$$\check{\widehat{f}}(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} \widehat{f}(\xi)\, d\xi$$

is only valid if we know that $\widehat{f}$ belongs to $L^1(\mathbb{R})$, which is generally not the case (we know that $\widehat{f} \in L^\infty(\mathbb{R})$).

In short – we may define the "inverse Fourier transform" in its own right for any integrable function, but we cannot say that it is the "inverse of the Fourier transform" in $L^1(\mathbb{R})$, in the sense that the composition of the two operators would constitute the identity.

The best approximation of such a result follows from the subsequent lemma, which we can prove by taking the analogous result for the Schwartz class and using the fact that $\mathcal{S}(\mathbb{R})$ is a dense subset of $L^1(\mathbb{R})$.

LEMMA 3.1. *For $f \in L^1(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} f(x)\widehat{g}(x)\, dx = \int_{\mathbb{R}} \widehat{f}(x) g(x)\, dx.$$

COROLLARY 3.2 (Fourier inversion on $L^1(\mathbb{R})$). *If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then $(\widehat{f})^{\vee} = f$ almost everywhere.*

PROOF. We may prove it in the same way as we did for the Schwartz class. In the previous lemma consider the heat kernel

$$g(x) = \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi|x-t|^2}{\varepsilon^2}}$$

and let $\varepsilon \to 0$ (see the proof of the next lemma, too). $\square$

Notice that adding *almost everywhere* is essential, since we already know that the Fourier transform (and in the same way also the inverse Fourier transform) "produces" continuous functions i.e. if we start with a function which is not continuous on a set of measure 0, the result will "smooth out" those discontinuities.

Observe that by the same token we may describe more properties of functions $f \in L^1(\mathbb{R})$ that have the chance of also satisfying the assumption $\widehat{f} \in L^1(\mathbb{R})$: they have to have a continuous representative in their class of abstraction $[f]$ and they have to be bounded, i.e. $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

In a similar way we may also get another result.

LEMMA 3.3. *If $f \in L^1(\mathbb{R})$ is continuous at 0, then*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} e^{-\pi\varepsilon\xi^2} \widehat{f}(\xi)\, d\xi = f(0).$$

PROOF. We have

$$\begin{aligned}\int_{\mathbb{R}} e^{-\pi\varepsilon\xi^2} \widehat{f}(\xi)\, d\xi &= \int_{\mathbb{R}} \widehat{e^{-\pi\varepsilon\xi^2}} f(x)\, dx \\ &= \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{\pi x^2}{\varepsilon^2}} f(x)\, dx = (h_\varepsilon * f)(0).\end{aligned}$$

Because h_ε is an approximate identity, the last term converges to $f(0)$ when $\varepsilon \to 0$, provided that f is continuous at 0. ☐

Finally, we are able to identify some of the "invertible" functions.

LEMMA 3.4. *If $f \in L^1(\mathbb{R})$ is continuous at 0 and $\widehat{f}$ is such that $\widehat{f} = \operatorname{Re}\widehat{f}$ and $\widehat{f} \geqslant 0$ then $\widehat{f} \in L^1(\mathbb{R})$, $\|\widehat{f}\|_1 = f(0)$ and $f = (\widehat{f})^\vee$ almost everywhere.*

PROOF. Because $\widehat{f} \geqslant 0$, we can use the previous lemma and the Fatou lemma to obtain

$$\begin{aligned}f(0) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}} e^{-\pi\varepsilon\xi^2} \widehat{f}(\xi)\, d\xi \\ &\geqslant \int_{\mathbb{R}} \liminf_{\varepsilon\to 0} e^{-\pi\varepsilon\xi^2} \widehat{f}(\xi)\, d\xi = \int_{\mathbb{R}} \widehat{f}(\xi)\, d\xi = \|\widehat{f}\|_1.\end{aligned}$$

Hence $\widehat{f} \in L^1(\mathbb{R})$. Now we can repeat this calculation using the Lebesgue dominated convergence theorem instead, with $\widehat{f}$ as the dominating function, since $e^{-\pi\varepsilon\xi^2} \leqslant 1$. Thus

$$f(0) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} e^{-\pi\varepsilon\xi^2} \widehat{f}(\xi)\, d\xi = \int_{\mathbb{R}} \lim_{\varepsilon\to 0} e^{-\pi\varepsilon\xi^2} \widehat{f}(\xi)\, d\xi = \|\widehat{f}\|_1.$$

The last statement follows from the result we discussed previously. ☐

4. Fourier transform in $L^2(\mathbb{R})$

Recall that on the torus we could prove that the Fourier transform is an isometry between $L^2(\mathbb{T})$ (we defined that space in a different but ultimately equivalent way) and the space of square-summable series $\ell^2(\mathbb{Z})$. We called it the Parseval identity. We also proved a similar result for functions in the Schwarz class.

However, for $p > 1$ and $f \in L^p(\mathbb{R})$, the integral in the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\, dx$$

cannot be computed. In particular, we are not able to use it directly to define the Fourier transform on $L^2(\mathbb{R})$.

We may, though, use the formula for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then we may use the fact that $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is a dense subspace (since, for example, it contains the Schwartz class, which is already a dense set) and extend the operator $f \mapsto \widehat{f}$ in an abstract way onto $L^2(\mathbb{R})$.

To this end we need the following lemma.

LEMMA 4.1. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\|\widehat{f}\|_2 = \|f\|_2$.*

PROOF. Let $h = f * \overline{\widetilde{f}}$, where $\widetilde{f}(x) = f(-x)$ and the bar indicates complex conjugation. Then $h \in L^1(\mathbb{R})$, $\widehat{h} = |\widehat{f}|^2 \geqslant 0$, and h is continuous at zero (see Problem 1). Therefore by Lemma 3.4

$$\|\widehat{f}\|_2^2 = \|\widehat{h}\|_1 = h(0) = \int_{\mathbb{R}} f(x)\overline{\widetilde{f}(-x)}\, dx = \|f\|_2^2. \qquad \square$$

This allows us to conclude that $f \mapsto \widehat{f}$ is an isometry on a dense subspace of $L^2(\mathbb{R})$, and hence has a unique extension onto $L^2(\mathbb{R})$. Once again, keep in mind that using the usual formula for the Fourier transform is in this case perilous. Notice as well that this procedure only works in $L^2(\mathbb{R})$ and not for any other $p > 1$.

Naturally, a similar story may be told about the inverse Fourier transform, and those two operators are now actual inverses of one another on $L^2(\mathbb{R})$.

5. Fourer transform in-between $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$

The following two lemmas provide a basis for so-called interpolation. These results depend heavily on the knowledge of complex analysis and we only use them to prove a rather technical (albeit important) result regarding the Fourier transform on spaces "between" $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ (the Hausdorff-Young inequality).

It is therefore only recommended to work through the proofs of those lemmas if you are familiar and comfortable with the results from complex analysis *and* you are interested in deeper understanding of the problems in Fourier analysis from the theoretical perspective.

For everyone else it is still advised to read the statements of all the lemmas, and to read the proof of the Hausdorff-Young inequality by simply assuming that the Riesz-Thorin lemma is valid.

Lemma 5.1 (Hadamard three-lines lemma). *Let $F : \mathbb{C} \to \mathbb{C}$ be an analytic function in the open strip*

$$S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\},$$

continuous and bounded on $\operatorname{cl} S$ *and such that*

$$|F(z)| \leqslant B_0 \quad \textit{when } \operatorname{Re} z = 0,$$
$$|F(z)| \leqslant B_1 \quad \textit{when } \operatorname{Re} z = 1,$$

where $0 < B_0, B_1 < \infty$. *Then*

$$|F(z)| \leqslant B_0^{1-\theta} B_1^{\theta}$$

when $\operatorname{Re} z = \theta$, *for any* $0 \leqslant \theta \leqslant 1$.

Proof. Define analytic functions

$$G(z) = \frac{F(z)}{B_0^{1-z} B_1^{z}} \quad \text{and} \quad G_n(z) = G(z) e^{(z^2-1)/n}.$$

Since F is bounded on the closed unit strip and $B_0^{1-z} B_1^{z}$ is bounded from below, we conclude that G is bounded by some constant M on the closed strip $\operatorname{cl} S$. We note that G is bounded by 1 on the boundary of S. Since

$$|G_n(x+iy)| \leqslant M e^{-y^2/n} e^{(x^2-1)/n} \leqslant M e^{-y^2/n},$$

we deduce that $G_n(x+iy)$ converges to zero uniformly in $0 \leqslant x \leqslant 1$ as $|y| \to \infty$. Select $y_n > 0$ such that for $|y| \geqslant y_n$, $|G_n(x+iy)| \leqslant 1$ uniformly in $x \in [0,1]$.

By the maximum principle we obtain that $|G_n(z)| \leqslant 1$ in the rectangle $[0,1] \times [-y_n, y_n]$. Hence $|G_n(z)| \leqslant 1$ everywhere in the closed strip. Letting $n \to \infty$, we conclude that $|G(z)| \leqslant 1$ in the closed strip $\operatorname{cl} S$, and the expected result for the function F follows immediately. □

Lemma 5.2 (Riesz-Thorin interpolation theorem). *Let (X, μ) and (Y, ν) be two measure spaces. Let T be a linear operator defined on the set of all simple functions on X and taking values in the set of measurable functions on Y. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and assume that*

$$\|T(f)\|_{q_0} \leqslant M_0 \|f\|_{p_0}$$
$$\|T(f)\|_{q_1} \leqslant M_1 \|f\|_{p_1},$$

for all simple functions f on X. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_q \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_p$$

for all simple functions f on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

REMARK 5.3. *The inequalities we impose on the operator T, combined with the fact that simple functions are dense in L^p-spaces, mean simply that T is a bounded operator from $L^{p_0}(X,\mu)$ to $L^{q_0}(Y,\nu)$, and at the same time from $L^{p_1}(X,\mu)$ to $L^{q_1}(Y,\nu)$. The lemma then says that T has a unique extension as a bounded operator from $L^p(X,\mu)$ to $L^q(Y,\nu)$ for all p and q which for some θ satisfy the given identities.*

Results of these type are called interpolation theorems*, since they provide a way of understanding the behaviour of a given operator on spaces lying "between" two other "extreme" spaces where that behaviour is known. Notice that since L^p-spaces are usually not comparable, the quotation marks are necessary (at least in this naive formulation).*

PROOF. Let

$$f = \sum_{k=1}^{m} a_k e^{i\alpha_k} \mathbb{1}_{A_k}$$

be a simple function on X, where $a_k > 0$, α_k are real, and A_k are pairwise disjoint subsets of X with finite measure. We need to control

$$\|T(f)\|_q = \sup\left\{\left|\int_Y T(f) g\, d\nu\right| : g \in L^{q'}(\nu),\ \|g\|_{q'} \leqslant 1\right\},$$

where, because of the definition of the integral, we may in fact consider only simple functions g on Y (with the q'-norm bounded by 1). Let

$$g = \sum_{j=1}^{n} b_j e^{i\beta_j} \mathbb{1}_{B_j},$$

where $b_j > 0$, β_j are real, and B_j are pairwise disjoint subsets of Y with finite measure (f and g are now two fixed simple functions).

Let

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1} z \quad \text{and} \quad Q(z) = \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1} z.$$

For z in the closed strip $\operatorname{cl} S = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Re} z \leqslant 1\}$, define

$$F(z) = \int_Y T(f_z) g_z\, d\nu,$$

where

$$f_z = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \mathbb{1}_{A_k}, \quad g_z = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \mathbb{1}_{B_j}.$$

By linearity of the integral and the operator T we get

$$F(z) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_j} \int_Y T(\mathbb{1}_{A_k}) \mathbb{1}_{B_j}\, d\nu,$$

and hence F is analytic in z, since $a_k, b_j > 0$.

Take $z \in S$ with $\operatorname{Re} z = 0$. Because the sets A_k are disjoint, we have

$$\|f_z\|_{p_0}^{p_0} = \|f\|_p^p, \quad \text{since} \quad \left|a_k^{P(z)}\right| = a_k^{p/p_0}.$$

Similarly, by the disjointness of the sets B_j we notice that

$$\|g_z\|_{q_0'}^{q_0'} = \|f\|_{q'}^{q'}, \quad \text{since} \quad \left|b_j^{Q(z)}\right| = b_j^{q'/q_0'}.$$

In the same way, when $\operatorname{Re} z = 1$ we obtain

$$\|f_z\|_{p_1}^{p_1} = \|f\|_p^p \quad \text{and} \quad \|g_z\|_{q_1'}^{q_1'} = \|f\|_{q'}^{q'}.$$

For $\operatorname{Re} z = 0$, the Hölder inequality and the hypothesis now give us

$$|F(z)| \leqslant \|T(f_z)\|_{q_0}\|g_z\|_{q_0'} \leqslant M_0\|f_z\|_{p_0}\|g_z\|_{q_0'} = M_0\|f\|_p^{p/p_0}\|g\|_{q'}^{q'/q_0'}.$$

Similarly, for $\operatorname{Re} z = 1$ we obtain

$$|F(z)| \leqslant M_1\|f\|_p^{p/p_1}\|g\|_{q'}^{q'/q_1'}.$$

We observe that F is analytic in the open strip S and continuous on its closure. Also, F is bounded on the closed unit strip (by some constant that depends on f and g). Therefore, by the Hadamard three-line lemma we obtain

$$|F(z)| \leqslant \left(M_0\|f\|_p^{p/p_0}\|g\|_{q'}^{q'/q_0'}\right)^{1-\theta}\left(M_1\|f\|_p^{p/p_1}\|g\|_{q'}^{q'/q_1'}\right)^{\theta} = M_0^{1-\theta}M_1^{\theta}\|f\|_p^{r_1}\|g\|_q^{r_2},$$

when $\operatorname{Re} z = \theta$ and $\theta \in [0,1]$. Notice that by assumption and the definition of P we have

$$r_1 = P(\theta) = \frac{p(1-\theta)}{p_0} + \frac{p\theta}{p_1} = 1$$

and $r_2 = Q(\theta) = 1$ as well. Hence $f_\theta = f$, $g_\theta = g$ and

$$F(\theta) = \int_Y T(f)g\, d\nu.$$

Taking the supremum over simple functions g on Y such that $\|g\|_{q'} \leqslant 1$, we obtain the result. □

COROLLARY 5.4 (Hausdorff-Young inequality). *If $f \in L^p(\mathbb{R})$ for $1 \leqslant p \leqslant 2$ then for q such that $1 = \frac{1}{p} + \frac{1}{q}$ we have $\|\widehat{f}\|_q \leqslant \|f\|_p$ (when $p = 1$ then we take $q = \infty$).*

PROOF. We apply the Riesz-Thorin lemma to the linear operator $f \mapsto \widehat{f}$, interpolated between $\|\widehat{f}\|_\infty \leqslant \|f\|_1$ and $\|\widehat{f}\|_2 = \|f\|_2$. □

As another consequence of the Riesz-Thorin lemma we obtain the following useful inequality for convolutions. It can also be proved directly, but now we get it almost immediately.

Corollary 5.5 (Young inequality for convolutions). *Suppose that* $1 \leqslant p, q, r \leqslant \infty$ *satisfy* $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. *If* $f \in L^p(\mathbb{R})$ *and* $g \in L^q(\mathbb{R})$ *then* $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$.

Proof. As the linear operator in the Riesz-Thorin lemma we consider the convolution $T(f) = f * g$ for a fixed function g. Then we observe that $T : L^1(\mathbb{R}) \to L^q(\mathbb{R})$ and $T : L^{q'}(\mathbb{R}) \to L^\infty(\mathbb{R})$, where $1 = \frac{1}{q} + \frac{1}{q'}$ and we get the result by interpolation (see Problem 3). □

6. Convergence and summability of the Fourier transform

Since the direct inverse transform is not always available, we may also consider the question of recovering a function from its Fourier transform in a similar way as in the case of Fourier series. Let us only have a brief look at some of the basic results in order to compare and contrast them to the theory we developed for the Fourier series.

We would like to determine if and when

$$\lim_{R\to\infty} \int_{-R}^{R} e^{2\pi i x\xi} \widehat{f}(\xi)\, d\xi = f(x),$$

where the limit can be taken in $L^p(\mathbb{R})$ or pointwise almost everywhere. Let the partial sum operator S_R be defined by

$$\widehat{S_R f} = \mathbb{1}_{[-R,R]} \widehat{f}.$$

Then we can write the same question as

$$\lim_{R\to\infty} S_R f = f \quad \text{(for an appropriate notion of the limit)}.$$

It turns out that a necessary and sufficient condition for the convergence in norm is that $\|S_R f\|_p \leqslant C_p \|f\|_p$, where C_p is independent of R. This is in fact the case, at least in dimension 1 (i.e. on $\mathbb{R}$), which is the only case we consider here. In higher dimensions, as a general rule, there is no convergence in norm when $p \neq 2$, but there are partial results available.

On $\mathbb{R}$ we have $S_R f(x) = D_R * f(x)$, where D_R is the Dirichlet kernel,

$$D_R(x) = \int_{-R}^{R} e^{2\pi i x\xi}\, d\xi = \frac{\sin(2\pi R x)}{\pi x}.$$

This function is not integrable, but it belongs to $L^q(\mathbb{R})$ for every $q > 1$, hence $D_R * f$ is well-defined if $f \in L^p(\mathbb{R})$ for some $1 < p < \infty$.

Almost everywhere convergence depends on the bound

$$\Big\| \sup_R |S_R f| \Big\|_p \leqslant C_p \|f\|_p.$$

This holds if $1 < p < \infty$ (the Carleson-Hunt theorem).

For the Fourier transform, the method of Cesàro summability consists in taking integral averages of the partial sum operators,

$$\sigma_R f(x) = \frac{1}{R}\int_0^R S_t f(x)\,dt$$

and determining if $\lim_{R\to\infty} \sigma_R f(x) = f(x)$. On $\mathbb{R}$ we have

$$\sigma_R f(x) = F_R * f(x),$$

where F_R is the Fejér kernel,

$$F_R(x) = \frac{1}{R}\int_0^R D_t(x)\,dt = \frac{\sin^2(\pi R x)}{R(\pi x)^2}.$$

The Fejér kernel is integrable. It can be proved that for $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{R})$ we have

$$\lim_{R\to\infty} F_R * f = f.$$

Questions:

- Can you verify at least some of the statements about density of sets in $L^p(\mathbb{R})$ spaces from the first section?
- Can you find examples of functions to show that
$$L^1(\mathbb{R})\backslash L^p(\mathbb{R}) \neq \varnothing \quad \text{and} \quad L^p(\mathbb{R})\backslash L^1(\mathbb{R}) \neq \varnothing$$
for every $p > 1$, or at least for $p = \infty$?
- Can you recall the formulation of the Riemann-Lebesgue lemma for the Fourier **series** and write its proof again, now knowing the proper definition of $L^1(\mathbb{T})$?
- Do you understand the problem with inverting the Fourier transform in $L^1(\mathbb{R})$?
- Can you verify that the extension of the Fourier transform from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ exists and is unique?
- Can you justify the name of Hadamard's lemma?
- Can you see similarities between the Dirichlet kernels D_R defined on $\mathbb{R}$ and D_N defined on $\mathbb{T}$?

Problems:

PROBLEM 1. Let $f \in L^1(\mathbb{R})$ and $h = f * \overline{\widetilde{f}}$, where $\widetilde{f}(x) = f(-x)$ and the bar indicates complex conjugation. Show that h is continuous at 0.

PROBLEM 2. Let $1 < p < r < q \leqslant \infty$ be such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$ for some $\theta \in [0,1]$ and suppose that $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$. Show that
$$\|f\|_r \leqslant \|f\|_p^{\theta}\|f\|_q^{1-\theta}.$$
Conclude that if f is in $L^p(\mathbb{R})$ and f is in $L^q(\mathbb{R})$, then f is in $L^r(\mathbb{R})$ for all $r \in [p,q]$. Hint: Use the Hölder inequality.

PROBLEM 3. Fill the gaps in the proof of the Young inequality for convolutions.

PROBLEM 4. Verify the formula defining the Dirichlet kernel
$$D_R(x) = \int_{-R}^{R} e^{2\pi i x\xi}\, d\xi = \frac{\sin(2\pi R x)}{\pi x}.$$
and show that for every $1 < p < \infty$ and $f \in L^p(\mathbb{R})$ we have
$$\lim_{R\to\infty} \|D_R * f - f\|_p = 0.$$

PROBLEM 5. Verify the formula defining the Fejér kernel
$$F_R(x) = \frac{1}{R}\int_0^R D_t(x)\, dt = \frac{\sin^2(\pi R x)}{R(\pi x)^2}.$$
and show that for every $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{R})$ we have
$$\lim_{R\to\infty} \|F_R * f - f\|_p = 0.$$