FOURIER ANALYSIS

PROBLEM LIST 1

Problem 1. The modulus (magnitude) of a complex number is defined as $|x + iy| = \sqrt{x^2 + y^2}$ Show that $|e^z| \leqslant e^{|z|}$.

Problem 2. Using definitions of e^z, $\sin(z)$ and $\cos(z)$ verify that $e^{x+iy} = e^x(i\sin(y) + \cos(y))$ for every $x, y \in \mathbb{R}$ and $|e^z| = e^{\mathrm{Re}(z)}$.

Problem 3. Using the Euler formulas prove the following identities

$$\begin{aligned} 2\sin\psi\sin\theta &= \cos(\psi-\theta) - \cos(\psi+\theta), \\ 2\sin\psi\cos\theta &= \sin(\psi-\theta) + \sin(\psi+\theta), \\ 2\cos\psi\cos\theta &= \cos(\psi-\theta) + \cos(\psi+\theta). \end{aligned}$$

Problem 4. Show that if a function f is continuous and periodic, and is not constant, then it has a basic period.

Problem 5. Show that the following function

$$f(x) = \begin{cases} 1 & \text{when } x \in \mathbb{Q}, \\ 0 & \text{when } x \notin \mathbb{Q} \end{cases}$$

does not have a basic period.

Definition. The following infinite set of functions

$$\{\sin nx, \cos nx\}_{n\in\mathbb{N}_0} = \{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$$

is called the trigonometric system.

Recall the two formulas describing trigonometric polynomials

$$W(x) = a_0 + \sum_{k=1}^{N} \big(a_k\cos(kx) + b_k\sin(kx)\big), \tag{1}$$

$$W(x) = \sum_{k=-N}^{N} c_k e^{ikx}, \qquad \{c_k\} \subset \mathbb{C}. \tag{2}$$

Problem 6. Prove that the trigonometric system is an orthogonal system and that $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ is orthonormal.

Problem 7. Show that expressions (1) and (2) are equivalent, i.e. for every pair of sets $\{a_k\}, \{b_k\} \subset \mathbb{C}$ there exists a set $\{c_k\} \subset \mathbb{C}$ such that they describe the same function W, and vice-versa. How about only real coefficients?

Problem 8. Using Problem 6, write a formula for coefficients a_n and b_n of a trigonometric polynomial in the form (1).

Problem 9. How to adjust the definition of trigonometric polynomials if instead of 2π–periodic functions we are interested in 1–periodic functions? How about an arbitrary period?