FOURIER ANALYSIS

PROBLEM LIST 2

Problem 1. Construct a sequence of polynomials which approximates $y \mapsto |y|$. Hint: Write $|y|$ as $\sqrt{1+(y^2-1)}$ and consider its Taylor expansion.

Problem 2. Let $\mathcal{A}_+ = \text{lin}\{e^{inx} : n \geqslant 0\} \subset C(\mathbb{T};\mathbb{C})$. Show that $\mathcal{A}_+$ is an algebra which separates points and does not vanish in $\mathbb{T}$, but it is not dense in $C(\mathbb{T})$.

Problem 3. Consider $C_{\mathbb{R}}\big([0,1]\times[0,1]\big)$ and

$$\mathcal{A} = \text{lin}_{\mathbb{R}}\{f(x)g(y) : f,g \in C_{\mathbb{R}}\big([0,1]\big)\}$$

(the family $\mathcal{A}$ consist of (real) linear combinations of functions of separated variables). Show that $\mathcal{A}$ is dense in $C_{\mathbb{R}}\big([0,1]\times[0,1]\big)$.

Problem 4. Suppose that $\mathcal{A}$ is an algebra in $C_{\mathbb{R}}(K)$ which separates points. Prove that one of the following statements is always true:

(1) $\text{cl}\,\mathcal{A} = C_{\mathbb{R}}(K)$.
(2) There exists a point $x_0 \in K$ such that $\mathcal{A} = \{f \in C_{\mathbb{R}}(K) : f(x_0) = 0\}$.

Recall the two formulas describing trigonometric polynomials

$$(1) \qquad W(x) = a_0 + \sum_{k=1}^{N} \big(a_k \cos(kx) + b_k \sin(kx)\big),$$

$$(2) \qquad W(x) = \sum_{k=-N}^{N} c_k e^{ikx}, \qquad \{c_k\} \subset \mathbb{C}.$$

Problem 5. Show that functions of the form (1), where a_k and b_k are assumed to be real, constitute an algebra.

Problem 6. Show that functions of the form (2) constitute a complex self-adjoint algebra.

Problem 7. Show that complex trigonometric polynomials are dense in the space of continuous 2π-periodic complex functions, i.e. $C(\mathbb{T})$. Similarly, show that functions given by formula (1), where $a_k, b_k \in \mathbb{R}$ are dense in $C_{\mathbb{R}}(\mathbb{T})$