FOURIER ANALYSIS

PROBLEM LIST 3

Problem 1. Suppose $\widehat{f}(n)$ for $n \in \mathbb{Z}$ are the Fourier coefficients of a function f. Let $m \in \mathbb{N}$ and $g(x) = f(x)e^{imx}$. Express the Fourier coefficients of the function g by using the coefficients of the function f.

Problem 2. Let f be a differentiable function. Define $g(x) = f'(x)$. Express the Fourier coefficients of the function g by using the coefficients of the function f.

Problem 3. Show that

$$\int_{\mathbb{T}} f(x)\overline{g(x)}\,dx = 2\pi \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}.$$

Hint: Use the Parseval identity and the polarization formula for the inner product.

Problem 4. Show that $\widehat{f * g}(n) = 2\pi \widehat{f}(n)\widehat{g}(n)$.

Problem 5. Let $f(x) = |x|$ on $[-\pi, \pi]$. Show that

$$\widehat{f}(n) = \begin{cases} \dfrac{\pi}{2} & \text{for } n = 0, \\ \dfrac{-1 + (-1)^n}{\pi n^2} & \text{for } n \neq 0. \end{cases}$$

Problem 6. Using the result of the previous problem, show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 7. Prove that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. Hint: Use the formula which defines the Dirichlet kernel and notice that the function $\frac{1}{\sin t/2} - \frac{2}{t}$ is continuous on $(-\pi, \pi)$.