FOURIER ANALYSIS

PROBLEM LIST 5

Problem 1. A set $U \subset \mathbb{R}$ is open if for every $x \in U$ there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset U$. Prove that every open set may be represented as $U = \bigcup_{n=1}^{\infty}(p_n, q_n)$, where $p_n, q_n \in \mathbb{Q}$.

Problem 2. Let

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k.$$

Show that if $\mathcal{A}$ is a σ-field and $A_1, A_2, \ldots \in \mathcal{A}$ then $\limsup_{n\to\infty} A_n \in \mathcal{A}$ and $\liminf_{n\to\infty} A_n \in \mathcal{A}$.

Problem 3. Show that the σ–field $\mathrm{Bor}(\mathbb{R})$ is generated by each of the following families

(1) open intervals with rational ends;
(2) closed intervals;
(3) closed intervals with rational ends;
(4) half-lines $(-\infty, a]$;
(5) half-lines (a, ∞);
(6) half-lines with a rational end.

Problem 4. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-field of subsets of X and let $E \in \mathcal{A}$. Consider the family

$$\mathcal{A}_E = \{A \cap E : A \in \mathcal{A}\}$$

Show that $\mathcal{A}_E \subset \mathcal{P}(E)$ is a σ-field of subsets of E.

Problem 5. Show that if $\mathcal{A}$ is an *infinite* σ–field then $\mathcal{A}$ has at least $\mathfrak{c}$ elements. Hint: Show that in every infinite σ-field there exists a sequence of non-empty pairwise disjoint sets; recall that $\mathfrak{c}$ is the cardinality of $\mathcal{P}(\mathbb{N})$.

Problem 6. Show that if $\mathcal{A}_\alpha$ are σ-fields, then $\bigcap_\alpha \mathcal{A}_\alpha$ is a σ-field. Can we say the same about unions?