FOURIER ANALYSIS

PROBLEM LIST 6

Problem 1. Let (X, Σ, μ) be a measure space. Show that

$$E \approx F \Leftrightarrow \mu(E \backslash F) = \mu(F \backslash E) = 0$$

is an equivalence relation between sets in Σ. Show that

$$f \approx g \Leftrightarrow \mu\big(\{x : f(x) \neq g(x)\}\big) = 0$$

is an equivalence relation between Σ-measurable functions.

On $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}))$ we define the Dirac delta measure of the point $x \in \mathbb{R}$ by

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Problem 2. Show that for $f, g \in \mathcal{S}(\mathbb{R})$ we have $f * g \in \mathcal{S}(\mathbb{R})$ and for $f, g \in L^1(\mathbb{R})$ we have $f * g \in L^1(\mathbb{R})$

Problem 3. For a measure μ on the space $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}))$ and an integrable function f we define

$$(f * \mu)(x) = \int_{\mathbb{R}} f(x - y)\, \mu(dy)$$

Show that we have $f * \delta_0 = f$, i.e. δ_0 is the identity element (in the algebraic sense) of the operation of convolution. Can you say how δ_0 is related to approximate identities?

Problem 4. Calculate $f * \delta_z$ for $z \in \mathbb{R}$.

Problem 5. Let μ be a finite measure and define

$$g(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\, \mu(dx).$$

Show that g is a bounded continuous function. Calculate g for $\mu = \delta_z$, where $z \in \mathbb{R}$. What happens for $z = 0$? Can you define $\widehat{f * \delta_z}$ for a given function f? What do you notice?