NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4170 Fourieranalyse**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: D: All printed and hand-written support material is allowed.

Other information:
Technical support: If you experience technical problems during the exam, contact Orakel support services (73 59 16 00) as soon as possible before the examination time expires. If you don't get through immediately, hold the line until your call is answered.
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Problem 1 Let f be the 2π-periodic function defined by $f(x) = x(x^2 - \pi^2)$ in $[-\pi, \pi]$.

a) Compute the n-Fourier coefficient of f, for all $n \in \mathbb{Z}$.

b) Prove that the Fourier series of f converges uniformly to f.

c) Prove that $\zeta(6) = \dfrac{\pi^6}{945}$, where ζ denotes the Riemann zeta-function.

Problem 2

a) Let $f(x) = (x^2 + 2x + 2)e^{-\pi x^2}$. Compute the Fourier transform of f.

b) Let $g : \mathbb{R} \to \mathbb{R}$ be the function such that

$$e^{\pi t^2} g(t) = \int_{-\infty}^{\infty} (x^2 + 2x + 2)e^{-2\pi x(x-t)}\, dx,$$

for all $t \in \mathbb{R}$. Compute the Fourier transform of g.

Problem 3 Let $m, n \geq 0$ be integers. Suppose that for any function $G \in C^\infty(\mathbb{R})$ with compact support the following inequality holds:

$$\sup_{\xi \in \mathbb{R}} \left| \xi^m \widehat{G}(\xi) \right| \leq \int_{-\infty}^{\infty} |G^{(n)}(x)|\, dx.$$

Prove that $m = n$. (The function $G^{(n)}$ denotes the n-th derivative of G)

Problem 4 Let $\mathcal{F}$ be the family of entire functions $f : \mathbb{C} \to \mathbb{C}$ of exponential type 2π such that $f \in \mathcal{M}(\mathbb{R})$, $f(x) \geq 0$ for all $x \in \mathbb{R}$ and $f(0) = 1$.

a) Prove that for any $f \in \mathcal{F}$ we have

$$\int_{-\infty}^{\infty} f(x)\, dx \geq 1.$$

b) Give an example of a function $f \in \mathcal{F}$ such that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1. \tag{1}$$

c) Prove that there exists a unique function $f \in \mathcal{F}$ such that (1) holds.

Problem 5 Does there exist a 2π-periodic integrable function f such that $0 \le f(x) \le \pi$ for all $x \in [-\pi, \pi]$, and

$$\widehat{f}(n) = \frac{(-1)^n}{\sqrt{1+n^2}},$$

for all $n \in \mathbb{Z}$?

- For $n \in \mathbb{Z}$, the n-Fourier coefficient of f is defined by:

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ixn}\, dx.$$

- $\mathcal{M}(\mathbb{R})$ denotes the family of moderate decrease functions.
- The Fourier transform of f is defined by:

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx,$$

  where $\xi \in \mathbb{R}$.
- $C^\infty(\mathbb{R})$ denotes the class of infinitely differentiable functions.