Fourier Analysis

Professor Andrés Chirre

Book: Fourier Analysis. An introduction, Stein and Shakarchi

Schedule: Tuesday 8:15-10:00
Thursday 10:15-12:00
Thursday 18:15-19:00

Exercises from the book and new problems on Wiki

Using Jamboard

Class 1: Introduction to Fourier series

$\sin(x)$; $\sin(x+2\pi) = \sin(x)$ $\forall x \in \mathbb{R}$

2π – periodic

$\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$ $\forall x, y \in \mathbb{R}$

$\cos(x+y) = \cos x \cos y - \sin x \cdot \sin y$ $\forall x, y \in \mathbb{R}$

$e^{ix} = \cos x + i \sin x$ $\forall x \in \mathbb{R}$

$e^{2\pi i} = \cos(2\pi) + i \underbrace{\sin(2\pi)}_{0} = \cos(2\pi) = \cos 0 = 1$

$(e^{2\pi i})^n = 1^n \Rightarrow e^{2\pi i n} = 1$, $\forall n \in \mathbb{Z}$

$e^{ix} = \cos x + i \sin x$ (+)

$e^{-ix} = \cos x - i \sin x$ (−)

$\frac{e^{ix} + e^{-ix}}{2} = \cos x$; $\frac{e^{ix} - e^{-ix}}{2i} = \sin x$

$$\boxed{\sin(x) = \frac{-1}{2i} \cdot e^{-ix} + \frac{1}{2i} e^{ix}}$$

$$\boxed{\cos(x) = \frac{1}{2} e^{-ix} + \frac{1}{2} e^{ix}}$$

$$\boxed{\cos(2x) = \frac{1}{2} e^{-2ix} + \frac{1}{2} e^{2ix}}$$

$\cos^2(x)$:

$\cos(2x) = \cos^2 x - \sin^2 x$

$= \cos^2 x - (1 - \cos^2 x)$

$\cos(2x) = 2\cos^2 x - 1$

$\Rightarrow \cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2}$

$$\boxed{\cos^2(x) = \frac{1}{4} e^{-2ix} + \frac{1}{2} + \frac{1}{4} e^{2ix}}$$

$$\sin(3x) = 3\sin(x) - 4\sin^3(x)$$

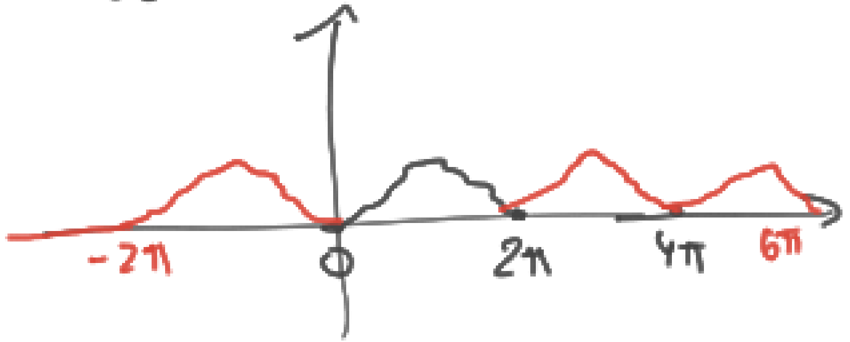
$$\sin^3(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$$

$$= -\frac{1}{4}\left(\frac{e^{3ix} - e^{-3ix}}{2i}\right) + \frac{3}{4}\left(\frac{e^{ix} - e^{-ix}}{2i}\right)$$

$$= \frac{1}{8i} e^{-3ix} - \frac{3}{8i} e^{-ix} + \frac{3}{8i} e^{ix} - \frac{1}{8i} e^{3ix}$$

$$f : [0, 2\pi] \to \mathbb{C}$$

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \ ?$$



$$= \sum_{n=-\infty}^{\infty} a_n (\cos(nx) + i\sin(nx))$$

$$= \sum_{n=-\infty}^{\infty} a_n \cos(nx) + i \sum_{n=-\infty}^{\infty} a_n \sin(nx)$$

F be a function defined on the circle

$$\{z \in \mathbb{C} : |z| = 1\} = \mathcal{C}$$

$$F : \mathcal{C} \to \mathbb{C}$$

$$z \in \mathcal{C} \Rightarrow z = e^{i\theta}, \quad \theta \in [0, 2\pi]$$

$$f : \mathbb{R} \to \mathbb{C}$$

$$f(\theta) = F(e^{i\theta})$$

$$f(\theta + 2\pi) = F(e^{i(\theta+2\pi)}) = F(e^{i\theta} \cdot \underbrace{e^{2\pi i}}_{1})$$

$$= F(e^{i\theta}) = f(\theta)$$

Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$.
The n^{th} Fourier coefficient of f is

$$\hat{f}(n)=\frac{1}{L}\int_a^b f(x)e^{\frac{-2\pi i n x}{L}}dx,$$

for $n\in\mathbb{Z}$ $\quad \boxed{L=b-a}$

$f:[0,2\pi]\to\mathbb{C}$

$$\hat{f}(n)=\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$$

$f:[-\pi,\pi]\to\mathbb{C}$

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$f(\theta)\sim\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{\frac{2\pi i n x}{L}}$$

$$f(\theta)\sim\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{inx}$$

$$\boxed{f(\theta)=\theta,\quad \theta\in[-\pi,\pi]}$$

$$f(\theta)\sim\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{inx}$$

$n\in\mathbb{Z}$

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\theta e^{-in\theta}d\theta$$

$$\int_{-\pi}^{\pi}\theta\, d\theta=\frac{\theta^2}{2}\Big|_{\theta=-\pi}^{\pi}$$

$$=\frac{\pi^2}{2}-\frac{(-\pi)^2}{2}$$

$$=\frac{\pi^2}{2}-\frac{\pi^2}{2}=0.$$

FUNDAMENTAL THEOREM OF CALCULUS:

$F : [a,b] \to \mathbb{R}$, F has continuous derivative

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

INTEGRATION BY PARTS:

$F, G : [a,b] \to R$; F, G have continuous derivatives

$$\underbrace{\int_a^b (FG)'}_{} = \int_a^b F' G + \int_a^b F G'$$

$$(FG)(x)\Big|_{x=a}^{x=b} = \int_a^b F'(x) G(x)\,dx + \int_a^b F(x) G'(x)\,dx$$

$$F(b)G(b) - F(a)G(a) = \int_a^b F'(x) G(x)\,dx + \int_a^b F(x) G'(x)\,dx$$

$$\boxed{\int_a^b F(x) G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x) G(x)\,dx}$$

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta e^{-in\theta}\,d\theta$$

$$\left(e^{-in\theta}\right)' = e^{-in\theta} \cdot (-in)$$

$$\left(\frac{e^{-in\theta}}{-in}\right)' = e^{-in\theta}$$

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta\,d\theta = \frac{1}{2\pi} \left\{ \frac{\theta^2}{2} \Big|_{\theta=-\pi}^{\theta=\pi} \right\}$$

$$\hat{f}(0) = 0$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\, e^{-in\theta}\, d\theta \quad ; \quad n \neq 0$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta \left(\frac{e^{-in\theta}}{-in}\right)' d\theta$$

$$= \frac{1}{2\pi}\left\{ \theta \cdot \frac{e^{-in\theta}}{-in}\Big|_{\theta=-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-in\theta}}{-in}\, d\theta \right\}$$

$$= \frac{1}{2\pi}\left\{ \pi\frac{e^{-in\pi}}{-in} - (-\pi)\frac{e^{in\pi}}{-in} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-in\theta}\, d\theta \right\}$$

$$= \frac{1}{2\pi}\left\{ -\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{1}{in}\int_{-\pi}^{\pi}\left(\frac{e^{-in\theta}}{-in}\right)' d\theta \right\}$$

$$= \frac{1}{2\pi}\left\{ -\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{1}{in}\left(\frac{e^{-in\theta}}{-in}\Big|_{\theta=-\pi}^{\pi} \right) \right\}$$

$$\hat{f}(n) = \frac{1}{2\pi}\left\{ -\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{e^{-in\pi}}{-i^2 n^2} - \frac{e^{in\pi}}{-i^2 n^2} \right\}$$

$$\hat{f}(n) = \frac{1}{2\pi}\left\{ -\frac{\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi} + \frac{e^{-in\pi}}{n^2} - \frac{e^{in\pi}}{n^2} \right\}$$

$$e^{in\pi} = \cos(n\pi) + i\,\sin(\cancel{n\pi}) \to 0$$

$$e^{in\pi} = (-1)^n \Leftrightarrow e^{-in\pi} = (-1)^n$$

$$\hat{f}(n) = \frac{1}{2\pi}\left\{ -\frac{\pi}{in}(-1)^n - \frac{\pi}{in}(-1)^n + \frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2} \right\}$$

$$\hat{f}(n) = \frac{1}{\cancel{2\pi}}\left\{ -\frac{\cancel{2\pi}}{in}(-1)^n \right\} = \frac{-(-1)^n}{in}$$

$$\hat{f}(n) = \frac{(-1)^{n+1}}{in}$$

$f: [-\pi, \pi] \longrightarrow \mathbb{R}$

$$f(\theta) = \theta$$

$\hat{f}(0) = 0$

$\hat{f}(n) = \dfrac{(-1)^{n+1}}{in}$; $\quad n \neq 0$

$$f(\theta) \sim \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n+1}}{in} \cdot e^{in\theta}$$

DIRICHLET TEST:

$\{a_n\} \subset \mathbb{R}$; $\{b_n\} \subset \mathbb{C}$

- $\{a_n\}$ decreases monotically to 0
- $\left|\sum_{n=1}^{N} b_n\right| \leq M$, for some $M > 1$ for all $N \geq 1$

$$\Rightarrow \sum_{n=1}^{\infty} a_n b_n \text{ is convergent}$$

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n+1} e^{in\theta}}{n} \quad ?$$

$$S_N(\theta) = \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{(-1)^{n+1}}{n} e^{in\theta}$$

$$= \sum_{n=-N}^{-1} \frac{(-1)^{n+1}}{n} e^{in\theta} + \underbrace{\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} e^{in\theta}}$$

$$= \sum_{n=1}^{N} \frac{(-1)^{-n+1} e^{-in\theta}}{-n} + \sum_{n=1}^{N}$$

$$= -\sum_{n=1}^{N} \frac{(-1)^{n+1} e^{-in\theta}}{n} + \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} e^{in\theta}$$

$$S_N(\theta) = -\sum_{n=1}^{N} \frac{(-1)^{n+1} e^{-in\theta}}{n} + \sum_{n=1}^{N} \frac{(-1)^{n+1} e^{in\theta}}{n}$$