Fourier analysis

Class 2:
Fourier series: examples and uniqueness

$f:[a,b]\to\mathbb{C}$ integrable function $\quad [\, f = u + i v$

$L = b-a$ $\qquad u:[a,b]\to\mathbb{R}$

$$\hat{f}(n) = \frac{1}{L}\int_a^b f(x)\, e^{-\frac{2\pi i n x}{L}}\,dx$$

$f:[-\pi,\pi]\to\mathbb{C}$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$$

$$f(\theta) \sim \underbrace{\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{inx}}$$

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)\, e^{inx}$$

$$\sum_{n\in\mathbb{Z}} a_n = \lim_{M\to\infty}\lim_{N\to\infty} \sum_{n=-M}^{N} a_n$$

$$\lim_{N\to\infty}\lim_{M\to\infty} \sum_{n=-M}^{N} a_n$$

if $\displaystyle\sum_{n=1}^{\infty} a_n$; $\displaystyle\sum_{n=-\infty}^{0} a_n$ are convergent

$$\Rightarrow \sum_{n\in\mathbb{Z}} a_n$$

$$g(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{inx} \quad \text{for some } x$$

$x\in[-\pi,\pi]$

$$\underbrace{\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{in(x+2\pi)}}_{g(x+2\pi)} = g(x)$$

$f : [-\pi, \pi] \to \mathbb{R}$

$f(\theta) = \theta$

$\hat{f}(n) = \begin{cases} \frac{(-1)^{n+1}}{in} ; & n \neq 0 \\ 0 & n = 0 \end{cases}$

$$\underbrace{f(\theta) \sim \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{(-1)^{n+1}}{in} e^{in\theta}}$$

$$S_N(f)(\theta) = \sum_{\substack{n=-N \\ n \neq 0}}^{N} \frac{(-1)^{n+1}}{in} e^{in\theta} \quad ; \quad N \geq 1$$

$$= \sum_{n=-N}^{-1} \frac{(-1)^{n+1}}{in} e^{in\theta} + \sum_{n=1}^{N} (-1)^{n+1} \frac{e^{in\theta}}{in}$$

$-N, \dots, -1$

$-n+1$

$$\sum_{n=1}^{N} (-1)^{-n+1} \frac{e^{i(-n)\theta}}{i(-n)} + \sum_{n=1}^{N} (-1)^{n+1} \frac{e^{in\theta}}{in}$$

$$= -\sum_{n=1}^{N} \frac{(-1)^{n+1}}{in} e^{-in\theta} + \underbrace{\sum_{n=1}^{N} (-1)^{n+1} \frac{e^{in\theta}}{in}}$$

$$A = \sum_{n=1}^{N} (-1)^{n+1} \frac{e^{in\theta}}{in} \quad ; \quad \text{Fix } \theta \in [-\pi, \pi]$$

DIRICHLET-TEST

$\{a_n\} \subset \mathbb{R}$; $\{b_n\} \subset \mathbb{C}$

i) $a_n \to 0$; decreases monotically

ii) $\left| \sum_{n=1}^{N} b_n \right| \leq M$, for some $M > 0$, $\forall N \geq 1$

$$\lim_{N \to \infty} \sum_{n=1}^{N} a_n b_n \text{ exist}$$

$a_n = \frac{1}{n}$ $\quad$ $b_n = \frac{(-1)^{n+1} e^{in\theta}}{i}$

$$b_n = \frac{(-1)^{n+1} e^{in\theta}}{i}$$

$$D = \sum_{n=1}^{N} \frac{(-1)^{n+1} e^{in\theta}}{i}$$

$$|D| \le \left| \sum_{n=1}^{N} \frac{(-1)^{n+1} e^{in\theta}}{i} \right| \le \sum_{n=1}^{N} \left| \frac{(-1)^{n+1} e^{in\theta}}{i} \right|$$

$$= \sum_{n=1}^{N} 1 = N$$

$$-e^{i\theta} D = D + (-1)^{N+2} \frac{e^{i(N+3)\theta}}{i} - \frac{e^{i\theta}}{i}$$

$$\frac{e^{i\theta}}{i} - (-1)^{N+2} \frac{e^{i(N+3)\theta}}{i} = D(1 + e^{i\theta})$$

$$\frac{e^{i\theta} - (-1)^{N+2} e^{i(N+3)\theta}}{i(1 + e^{i\theta})} = D \qquad e^{i\theta} \neq -1$$

$$D = \sum_{n=1}^{N} \frac{(-1)^{n+1} e^{in\theta}}{i} \Rightarrow (-1)e^{i\theta} D = \sum_{n=1}^{N} \frac{(-1)^{n+2} e^{i(n+1)\theta}}{i}$$

$$= \cancel{\frac{(-1)^3 e^{2i\theta}}{i}} + \cancel{\frac{(-1)^4 e^{3i\theta}}{i}} + \dots + \cancel{\frac{(-1)^{N+1} e^{iN\theta}}{i}} + (-1)^{N+2} \frac{e^{i(N+3)\theta}}{i}$$

$$+ \cancel{\frac{(-1)^2 e^{i\theta}}{i}} - \frac{(-1)^2 e^{i\theta}}{i}$$

$$(-1)e^{i\theta} D = D + (-1)^{N+2} e^{i(N+3)\theta}/i - e^{i\theta}/i$$

$$\frac{e^{i\theta} - (-1)^{N+2} e^{i(N+1)\theta}}{i(1+e^{i\theta})} = D \qquad e^{i\theta} = -1$$

$$|D| = \left| \frac{e^{i\theta} - (-1)^{N+2} e^{i(N+1)\theta}}{i(1+e^{i\theta})} \right| \leq |1| + |1|$$

$$|D| \leq \frac{2}{|1+e^{i\theta}|} \qquad \underline{\theta \in [-\pi, \pi]}$$

$$\Rightarrow \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{in\theta}}{in}}\ ; \quad \underline{e^{i\theta} = -1}$$

$$\cos\theta = -1$$

$$\boxed{\theta = \pi, \theta = -\pi}$$

$$\underline{\theta \in (-\pi, \pi)}$$

$$-\theta \in (-\pi, \pi)$$

$$-\sum_{n=1}^{N} \frac{(-1)^{n+1} e^{-in\theta}}{in} \ ? \checkmark \qquad \theta \in (-\pi, \pi)$$

$$S_N(f)(\theta) = -\sum_{n=1}^{N} \frac{(-1)^{n+1} e^{-in\theta}}{in} + \sum_{n=1}^{N} \frac{(-1)^{n+1} e^{in\theta}}{in}$$

$$\theta \in (-\pi, \pi)$$

$\lim_{N \to \infty} S_N(f)(\theta)$ exist

$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{(-1)^{n+1} e^{in\theta}}{in}$ is convergent for $\underline{\theta \in (-\pi, \pi)}$

$$S_N(f)(\theta) = -\sum_{n=1}^{N} (-1)^{n+1} \frac{e^{-in\theta}}{in} + \sum_{n=1}^{N} (-1)^{n+1} \frac{e^{in\theta}}{in}$$

$$(e^{i\theta})^n = (-1)^n$$

$$S_N(f)(\theta) = -\sum_{n=1}^{N} (-1)^{n+1} \frac{(-1)^{-n}}{in} + \sum_{n=1}^{N} (-1)^{n+1} \frac{(-1)^n}{in}$$

$$= -\sum_{n=1}^{N} \frac{(-1)}{in} + \sum_{n=1}^{N} \frac{-1}{in}$$

$$= \sum_{n=1}^{N} \frac{1}{in} - \sum_{n=1}^{N} \frac{1}{in} = 0$$

$\theta = \pi, -\pi$

$$S_N(f)(\theta) = 0 \Rightarrow \lim_{N\to\infty} S_N(f) = 0$$

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} (-1)^{n+1} \frac{e^{in\theta}}{in}, \quad \theta \in [-\pi, \pi]$$

is convergent

$P_N : [-\pi, \pi] \to \mathbb{C}$

$$P_N(\theta) = \sum_{k=-N}^{N} a_k e^{ik\theta} \; ; \quad a_N \neq 0 \text{ or } a_{-N} \neq 0$$

$$\widehat{P_N}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-N}^{N} a_k e^{ik\theta} \right) e^{-in\theta} \, d\theta$$

$$= \frac{1}{2\pi} \sum_{k=-N}^{N} a_k \int_{-\pi}^{\pi} e^{i(k-n)\theta} \, d\theta$$

$$\int_{-\pi}^{\pi} e^{im\theta}\,d\theta, \qquad m\in\mathbb{Z}$$

$$m=0 \Rightarrow \int_{-\pi}^{\pi} e^{im\theta}\,d\theta = 2\pi$$

$$m\neq 0 \Rightarrow \int_{-\pi}^{\pi} \left(\frac{e^{im\theta}}{im}\right)' d\theta = \frac{e^{im\theta}}{im}\Bigg|_{\theta=-\pi}^{\pi}$$

$$= \frac{e^{im\pi}}{im} - \frac{e^{-im\pi}}{im}$$

$$= \frac{(-1)^m}{im} - \frac{(-1)^m}{im} = 0$$

$$\widehat{P_N}(n) = \frac{1}{2\pi}\sum_{k=-N}^{N} a_k \int_{-\pi}^{\pi} e^{i\theta(k-n)}\,d\theta$$

$$= \frac{1}{2\pi}\, a_n \underbrace{\int_{-\pi}^{\pi} e^{0}\,d\theta}_{2\pi} + \frac{1}{2\pi}\sum_{\substack{k=-N\\ k\neq n}}^{N} a_k \underbrace{\int_{-\pi}^{\pi}\cdots}_{}\overset{0}{}$$

$$= a_n + 0 = a_n$$

$$\widehat{P_N}(n) = a_n$$

$$P_N(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$$

$$= \sum_{n=-N}^{N} \widehat{P_N}(n)\, e^{in\theta}$$

$$P_N(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta} \Rightarrow \widehat{P_N}(n) = a_n$$

$$|n| > N \Rightarrow a_n = 0$$

$$P_N(\theta) \sim \sum_{n\in\mathbb{Z}} \widehat{P}(n) e^{in\theta}$$

$$\sum_{n=-N}^{N} a_n e^{in\theta} = P_N(\theta)$$

$a_n = 1$

$$D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}$$

$$\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z}$$

$$\Rightarrow f \equiv 0$$

$f : [-\pi, \pi] \to \mathbb{C}$

f defined on the circle

$\Updownarrow$

$f : \mathbb{R} \to \mathbb{C}$ $\quad$ f is 2π- periodic

$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}$

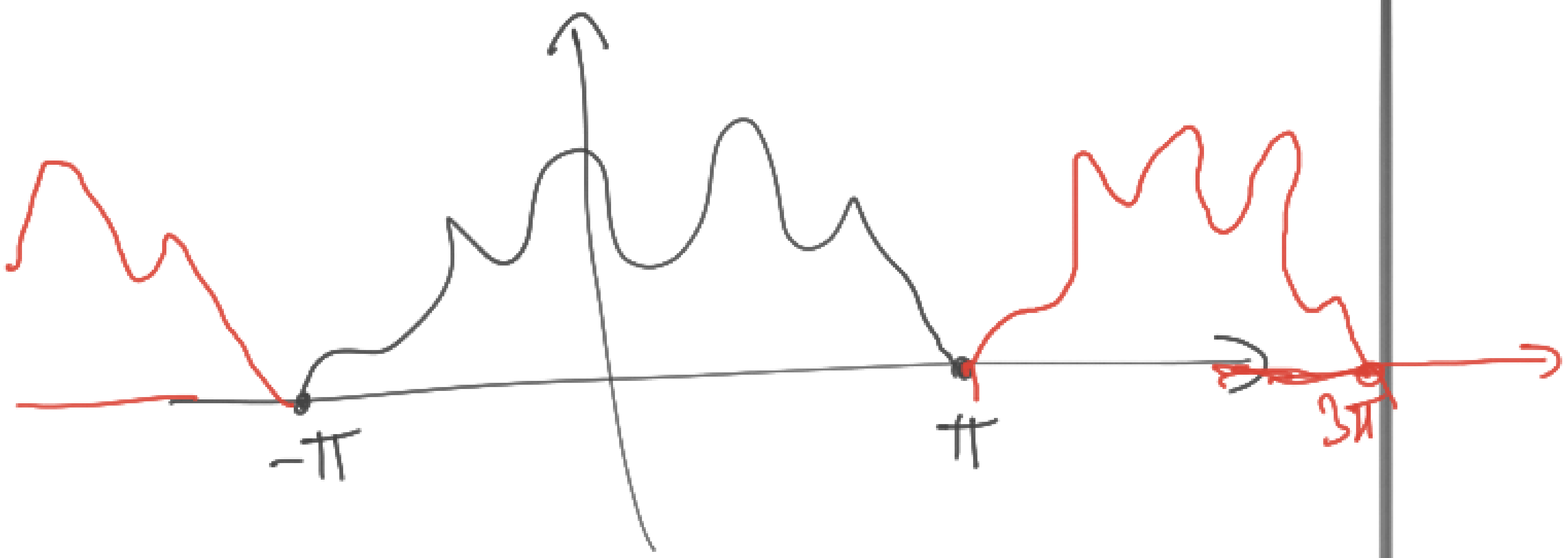
or

$f : [a,b] \to \mathbb{C}$; $\quad b - a = 2\pi$

$f(b) = f(a)$

$f : [-\pi, \pi] \longrightarrow \mathbb{C}$

$f(-\pi) = f(\pi)$



UNIQUENESS OF FOURIER SERIES

THEOREM: Let f be an integrable function defined on the circle. Suppose that $\hat{f}(n)=0 \quad \forall n \in \mathbb{Z}$

$\Rightarrow f(\theta_0)=0$; where θ_0 is a point of continuity of f.

Corollary 1: Let f is a continuous function defined on the circle. Suppose that $\hat{f}(n)=0 \quad \forall n \in \mathbb{Z}$

$$\Rightarrow f \equiv 0$$

Corollary 2: Let f and g be continuous function on the circle.

$\hat{f}(n)=\hat{g}(n) \quad \forall n \in \mathbb{Z}$

$$\Rightarrow f \equiv g$$

PROOF:

$h = f - g$; h cont. function on the circle

$\hat{h}(n) = \hat{f}(n) - \hat{g}(n)$

$\hat{h}(n)=0 \Rightarrow h \equiv 0$

$$f \equiv g$$

Corollary 3. Let f be a continuous function defined on the circle.
Suppose that: $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$

$$\Rightarrow S_N(f)(\theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}$$

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$$ uniformly on θ

$$f(\theta) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{in\theta}$$

$$f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{in\theta}$$

EXAMPLE: $0 \leq \theta \leq 2\pi$

$$f(\theta) = \frac{(\pi-\theta)^2}{4}.$$

$f(0) = \pi^2/4$

$f(2\pi) = \pi^2/4$

$$\hat{f}(n) = \begin{cases} \dfrac{(-1)^n e^{-in\pi}}{2n^2} ; & n \neq 0 \\ \dfrac{\pi^2}{12} ; & n = 0 \end{cases}$$

$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$?

$$\sum_{n=1}^{\infty} |\hat{f}(n)| < \infty \quad \sum_{n=-\infty}^{0} |\hat{f}(n)| < \infty$$

$$\sum_{n=1}^{N} |\hat{f}(n)|$$

$\boxed{a_n \geq 0}$, $\sum_{n=1}^{\infty} a_n$ is convergent?

$B_N = \sum_{n=1}^{N} a_n \Rightarrow B_1 \leq B_2 \leq B_3 \leq \ldots$

$\circ)$ $|B_N| \leq M$; for some M, $\forall N \geq 1$

$\Rightarrow \exists \lim_{N\to\infty} B_N$

$\exists \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \sum_{n=1}^{\infty} a_n$ ✓

$a_n \geq 0$; ✓ $\sum_{n=1}^{\infty} a_n$ is convergent?

$\sum_{n=1}^{N} a_n \leq M$ ✓

$$\sum_{n=1}^{N} |\hat{f}(n)| = \sum_{n=1}^{N} \left| \frac{\cancel{(-1)^n} \cancel{e^{-in\pi}}}{2n^2} \right|$$

$$= \sum_{n=1}^{N} \frac{1}{2n^2} = \frac{1}{2} \sum_{n=1}^{N} \frac{1}{n^2}$$

$$\sum_{n=1}^{N} \frac{1}{n^2}$$

$n \geq 2$:

$n-1 \leq x \leq n \Rightarrow \frac{1}{n} \leq \frac{1}{x}$

$\int_{n-1}^{n} \frac{dx}{n^2} \leq \int_{n-1}^{n} \frac{dx}{x^2}$; $\forall x \in [n-1, n]$

$\frac{1}{n^2} \leq \int_{n-1}^{n} \frac{dx}{x^2}$

$$\frac{1}{n^2} \leq \int_{n-1}^{n} \frac{dx}{x^2} \quad ; \quad \forall n \geq 2$$

$$\sum_{n=2}^{N} \frac{1}{n^2} \leq \sum_{n=2}^{N} \int_{n-1}^{n} \frac{dx}{x^2}$$

$$\sum_{n=2}^{N} \frac{1}{n^2} \leq \underbrace{\int_{1}^{N} \frac{dx}{x^2}}$$

$$\frac{x^{-1}}{-1} \Big|_{x=1}^{x=N}$$

$$1 + \sum_{n=2}^{N} \frac{1}{n^2} \leq 1 - \frac{1}{N} + 1$$

$$\sum_{n=1}^{N} \frac{1}{n^2} \leq 2 - \frac{1}{N} \leq 2$$