Class 4:
Uniqueness of Fourier Series

Let $f : [a,b] \to \mathbb{C}$ be an integrable function

$L = b - a.$

$$\hat{f}(n) = \frac{1}{L}\int_a^b f(x)\, e^{\frac{-2\pi i n x}{L}}\, dx, \quad n \in \mathbb{Z}$$

$f : [-\pi, \pi] \to \mathbb{C}$ integrable :

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \; ; \quad N \geq 1; \; x \in [-\pi, \pi]$$

$S_N(f)(x)$ converges ?

Theorem : Let $f : \mathbb{R} \to \mathbb{C}$ be a [illegible] 2π-periodic function. Suppose that ...... $\hat{f}(n)$ $\quad f(x+2\pi) = f(x)$

Theorem : Let $f : [-\pi, \pi] \to \mathbb{C}$ be a [illegible] function such that $f(-\pi) = f(\pi)$. Suppose that ...

Theorem : Let $f : [0, 2\pi] \to \mathbb{C}$ be a [illegible] function such that $f(0) = f(2\pi)$. Suppose that ...

Theorem : Let $f : [a, a+2\pi] \to \mathbb{C}$ be a [illegible] function $(a \in \mathbb{R})$; such that $f(a) = f(a+2\pi)$. Suppose that ....

integrable / continuous

Theorem : Let f be a function on the circle. Suppose that ...

$f: \text{---} \to \mathbb{C}$

$\circ$) WHICH INTERVAL IS DEFINED THE n^{th} FOURIER COEFFICIENT?

$f: \mathbb{R} \to \mathbb{C}$

$f: [0, 2\pi] \to \mathbb{C}$; $f(x) = f(x+2\pi)$

$f: [-\pi, \pi] \to \mathbb{C}$, $f(x) = f(x+2\pi)$

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\, dx \quad ✓$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \quad ✓$$

* INTEGRABLE FUNCTION ON THE CIRCLE.
* CONTINUOUS FUNCTION ON THE CIRCLE
* CONTINUOUSLY DIFFERENTIABLE ON THE CIRCLE.

Theorem: Let f be a function on the circle. Suppose that $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$. Then $f(\theta_0) = 0$, whenever f is continuous at the point θ_0.

PROOF:

(i) Assume that you had proved this theorem when g is real-valued.

Suppose $f: D \to \mathbb{C}$. $\quad f = u + iV$, $\quad \hat{f}(n) = 0$

$f(x) = u(x) + i V(x)$, where u, V are real-valued functions

$(\bar{f})(x) := \overline{f(x)} = u(x) - i V(x)$

$$u(x) = \frac{f(x) + \bar{f}(x)}{2}; \quad V(x) = \frac{f(x) - \bar{f}(x)}{2i}$$

$$\hat{u}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x) e^{-inx}\, dx = \frac{\hat{f}(n) + \hat{\bar{f}}(n)}{2}$$

$$\hat{\bar{f}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{f}(x) e^{-ixn} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\, e^{-ixn} dx$$

$$= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{ixn} dx}$$

$$= \overline{\hat{f}(-n)}$$

$$\boxed{\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}}$$

$$\hat{u}(n) = \frac{\hat{f}(n) + \hat{\bar{f}}(n)}{2}$$

$$\hat{u}(n) = \frac{\hat{f}(n) + \overline{\hat{f}(-n)}}{2}$$

$$\hat{u}(n) = 0 \quad \forall n \in \mathbb{Z}$$

$$\hat{v}(n) = 0 \quad \forall n \in \mathbb{Z}$$

$$u, v : [-\pi, \pi] \to \mathbb{R}$$

If f is continuous at the point θ_0

$\Rightarrow$ u, v are continuous at the point θ_0

$$u(\theta_0) = 0 \quad v(\theta_0) = 0 \Rightarrow f(\theta_0) = 0$$

Suppose that f is real-valued.

$f : [-\pi, \pi] \to \mathbb{R}$

We can assume that $\theta_0 = 0$ point of continuity

$g : [-\pi, \pi] \to \mathbb{R}$

$g(x) = f(x + \theta_1)$ $\hat{f}(n) = 0$

$\hat{g}(n) = 0$

• GOAL: $f(0) = 0$!

Suppose that $f(0) \neq 0 \Rightarrow$ $\boxed{f(0) > 0}$

$f(0) < 0$

$(-f)(0) > 0$

$$\boxed{f(0) > 0}$$

• $f : [-\pi, \pi] \to \mathbb{R}$; $f(-\pi) = f(\pi)$

• The function f is continuous at $\theta_0 = 0$.

• $\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z}$.

• $f(0) > 0$

$$\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z}$$

$$\boxed{\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = 0 \quad \forall n \in \mathbb{Z}}$$

If you remember; $P(x) = \underbrace{\sum_{k=-N}^{N} a_k e^{-ikx}}_{\text{Trigonometric polynomial}}$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) P(x)\, dx = 0 \text{; for any trigon. polynomial}$$

$$P(x) = \sum_{K=-N}^{N} a_K e^{-iKx}$$

$$\int_{-\pi}^{\pi} f(x) P(x)\, dx = \int_{-\pi}^{\pi} f(x) \left(\sum \ldots\right)$$

$$= \sum_{K=-N}^{N} a_K \underbrace{\int_{-\pi}^{\pi} f(x) e^{-iKx}\, dx}_{= 0}$$

$$\int_{-\pi}^{\pi} f(x) \cdot P(x)\, dx = 0$$

for any P: trigonometric polynomial

$f(0) > 0$, f is continuous at "0"

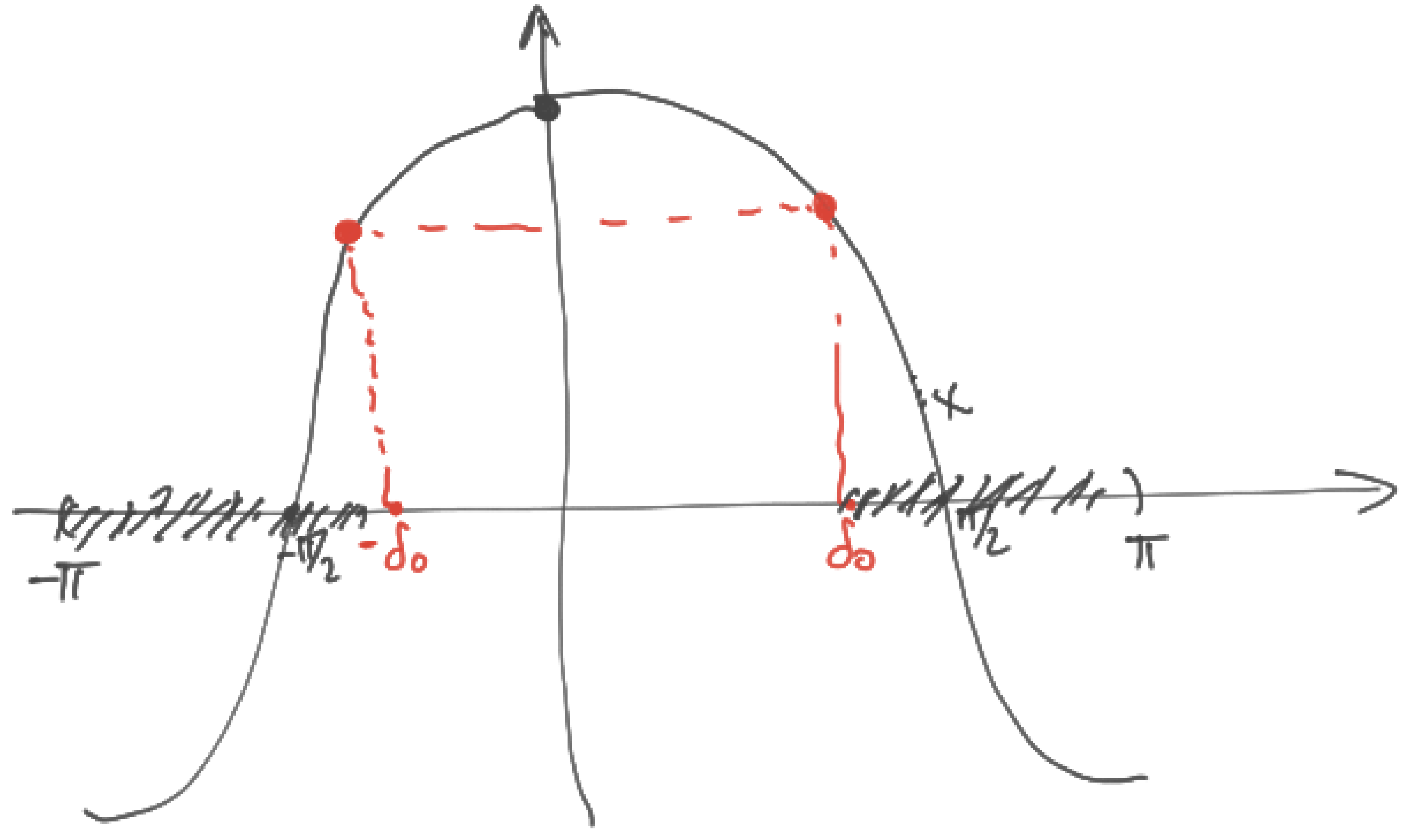
$$\exists\, \delta_0 > 0 :$$

$$|x| < \delta_0 \Rightarrow \underbrace{|f(x) - f(0)| < \frac{f(0)}{2}}$$

$$-\frac{f(0)}{2} < f(x) - f(0) < \frac{f(0)}{2}$$

$$|x| < \delta_0 \Rightarrow \underline{\frac{f(0)}{2} < f(x) < \frac{3f(0)}{2}}$$

$$\boxed{|x| < \delta_0 \Rightarrow f(x) > \frac{f(0)}{2}}$$



$$\delta_0 \le |X| \le \pi \Rightarrow \quad \cos x \le \cos \delta_0$$

$$1 - \cos x \geq \underbrace{1 - \cos \delta_0}_{\varepsilon_0}$$

$$1 - \frac{\varepsilon_0}{3} \geq \underbrace{\cos x + \frac{2\varepsilon_0}{3}}_{P(x)}$$

$$P(x) \le 1 - \frac{\varepsilon_0}{3}; \qquad \delta_0 \le |X| \le \pi$$

$$\underbrace{|P(x)| \le 1 - \frac{\varepsilon_0}{3};} \qquad \delta_0 \le |x| \le \pi$$

$$\Updownarrow$$

$$\frac{\varepsilon_0}{3} - 1 \le P(x) \le 1 - \frac{\varepsilon_0}{3}$$

$$\frac{\varepsilon_0}{3} \le P(x) + 1 \iff \frac{\varepsilon_0}{3} \le \cos x + \frac{2\varepsilon_0}{3} + 1$$

$$\iff -\frac{\varepsilon_0}{3} \le \cos x + 1$$

$$\Rightarrow \quad |P(x)| \le 1 - \frac{\varepsilon_0}{3}; \qquad \delta_0 \le |x| \le \pi$$

$$\varphi(x) = \cos x + 2\frac{\varepsilon_0}{3}$$

$$\lim_{x \to 0} \cos x = 1$$

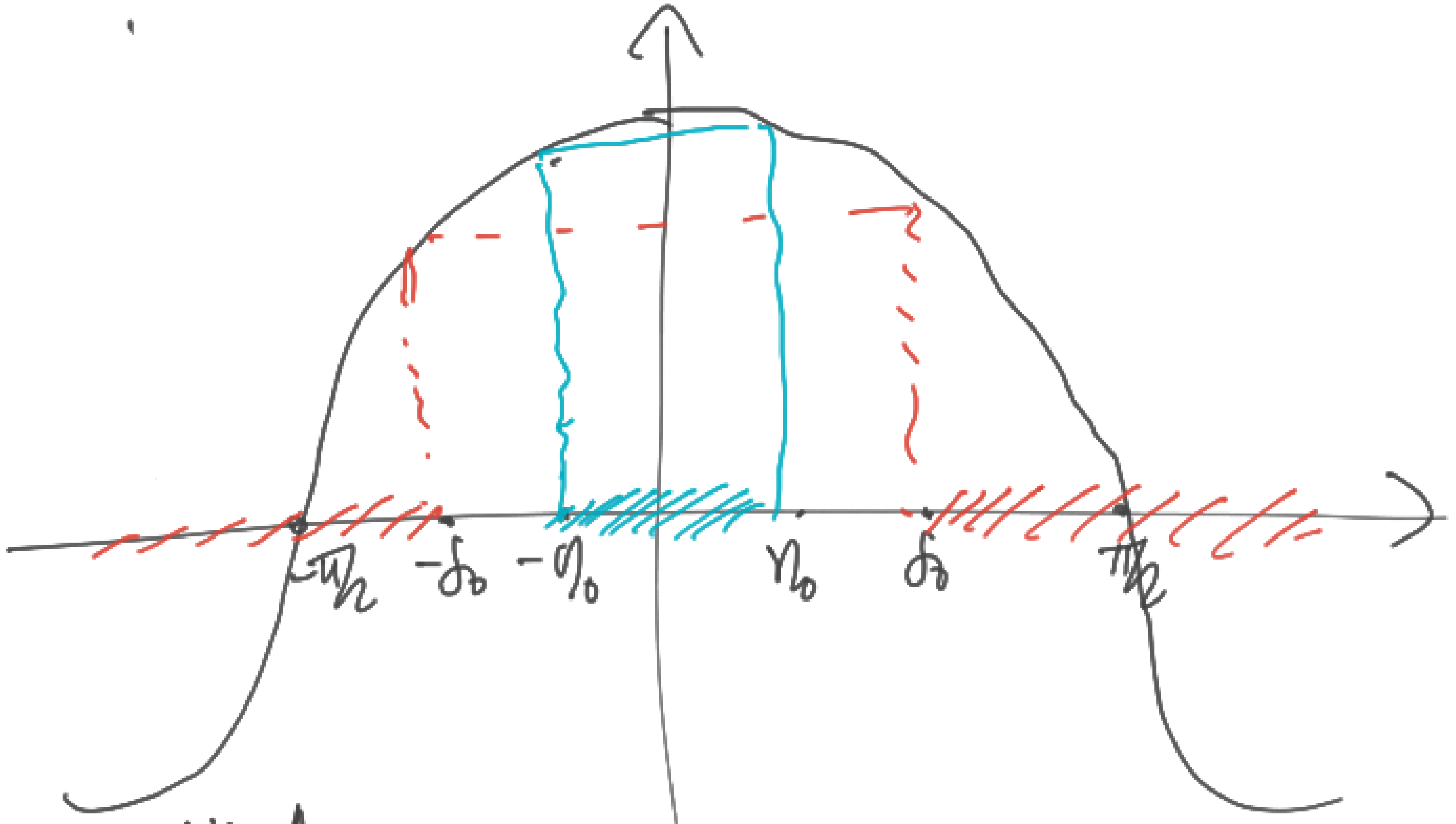
$$\exists\, \eta_0 > 0 : \quad |x| < \eta_0 < \delta_0$$

$$\Rightarrow \quad |\cos x - 1| < \frac{\varepsilon_0}{3}$$

$$1 - \cos x < \frac{\varepsilon_0}{3}$$

$$1 < \cos x + \frac{\varepsilon_0}{3}$$

$$|x| < \eta_0 \Rightarrow 1 + \frac{\varepsilon_0}{3} < \varphi(x)$$



$K \geq 1$

$$\varphi_K(x) = \left(\varphi(x)\right)^{2K}$$

$$\varphi_K(x) \geq 0$$

$$P(x) = 2\frac{\varepsilon_0}{3} + \cos\theta$$

$$P_k(x) = (P(x))^{2k}; \quad k \geq 1$$

$$\rightarrow |P_k(x)| \leq \left(1 - \frac{\varepsilon_0}{3}\right)^{2k}; \quad \delta_0 \leq |x| \leq \pi$$

$$P_k(x) > \left(1 + \frac{\varepsilon_0}{3}\right)^{2k}; \quad |x| \leq \eta_0 \leftarrow$$

$$\underline{P_k(x) \geq 0}; \quad \underline{\eta_0 \leq |x| \leq \delta_0}$$

$$(\eta_0 < \delta_0)$$

$$f(x) > \frac{f(0)}{2}; \quad |x| \leq \delta_0$$

$$\leftarrow \quad |x| \leq \eta_0$$

$$\underbrace{\int_{-\pi}^{\pi} f(x) P_k(x)\, dx = 0}$$

$$\int_{\delta_0 \leq |x| \leq \pi} f(x) P_k(x)\, dx + \int_{|x| \leq \eta_0} f(x) P_k(x)\, dx + \int_{\eta_0 \leq |x| \leq \delta_0} \overset{\geq 0}{f(x) P_k(x)}\, dx$$

$$I_1 + I_2 + I_3 = 0$$

$$I_3 \geq 0$$

$$I_2 = \int_{|x| \leq \eta_0} f(x) P_k(x)\, dx \geq \int_{|x| \leq \eta_0} \frac{f(0)}{2} P_k(x)\, dx$$

$$= \frac{f(0)}{2} \int_{|x| \leq \eta_0} P_k(x)\, dx \geq \frac{f(0)}{2} \left(1 + \frac{\varepsilon_0}{3}\right)^{2k} \int_{|x| \leq \eta_0} 1\, dx$$

$$I_2 \geq \frac{f(0)}{2} \left(1 + \frac{\varepsilon_0}{3}\right)^{2k} \cdot 2\eta_0$$

$$I_1 = \int_{\delta_0 \le |x| \le \pi} f(x) P_K(x)\,dx$$

$$|I_1| \le \int_{\delta_0 \le |x| \le \pi} |f(x)||P_K(x)|\,dx$$

$|f(x)| \le B,\quad \forall x \in [-\pi, \pi],\quad B > 0$

$$|I_1| \le B \int_{\delta_0 \le |x| \le \pi} |P_K(x)|\,dx$$

$$\le B.\left(1 - \frac{\varepsilon_0}{3}\right)^{2k} \underbrace{\int_{\delta_0 \le |x| \le \pi} 1\,dx}_{2\pi}$$

$$|I_1| \le B.\,2\pi \left(1 - \frac{\varepsilon_0}{3}\right)^{2k}$$

$$I_1 \geq -B.2\pi.\left(1 - \frac{\varepsilon_0}{3}\right)^{2k}$$

$$0 = \underline{I_1} + I_2 + I_3$$

$$\geq \underbrace{-B.2\pi\left(1 - \frac{\varepsilon_0}{3}\right)^{2k \to 0}}_{} + \underbrace{\frac{f(0)}{2}^{\to 0}\left(1 + \frac{\varepsilon_0}{3}\right)^{2k \to \infty}.2\eta_0^{\to 0}}_{}$$

K is free $\quad K \to \infty$

CONTRADICTION

$$\Rightarrow f(0) = 0$$

Corollary 1: If f, g are continuous function defined on the circle, and $\hat{f}(n) = \hat{g}(n)$ $\forall n \in \mathbb{Z}$

$\Rightarrow f \equiv g$

$h = f - g$

$\hat{h}(n) = 0 \quad \forall n \in \mathbb{Z}$

$\Rightarrow h \equiv 0$

Corollary: Suppose that f is continuous on the circle, and $\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$

$\Rightarrow S_N(f)(\theta) \to f(\theta)$; uniformly

PROOF: $f: [-\pi, \pi] \to \mathbb{C}$

$$S_N(f)(\theta) = \sum_{k=-N}^{N} \hat{f}(n) e^{in\theta}$$

$S_N(f)$ is a continuous function

$$|S_N(f)(\theta)| \le \sum_{k=-N}^{N} |\hat{f}(n)|$$

$$S_N(f)(\theta) \xrightarrow[N \to \infty]{} g(\theta)$$

uniformly

absolutely

Then g is continuous in $[-\pi, \pi]$

$$S_N(f)(-\pi) = S_N(f)(\pi)$$

$$\underset{N\to\infty}{} \quad g(-\pi) = g(\pi)$$

$$\hat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} S_N(f)(x)\, e^{-ikx}\, dx$$

uniform convergence

$$= \lim_{N\to\infty} \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} S_N(f)(x)\, e^{-ikx}\, dx}$$

$$= \lim_{N\to\infty} \widehat{S_N(f)}(k) \qquad S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

$$= \lim_{N\to\infty} \hat{f}(k) = \hat{f}(k)$$

$$\hat{g}(k) = \hat{f}(k) \quad \forall k \in \mathbb{Z}$$

continuous continuous $\Rightarrow g \equiv f$

$$\boxed{\lim_{N\to\infty} S_N(f)(\cancel{x}) = f(x)}$$

EXAMPLE:

$$f(\theta) = \frac{(\pi-\theta)^2}{4}; \quad 0 \le \theta \le 2\pi$$

$f(0) = \pi^2/4$

$f(2\pi) = \pi^2/4$

$$\hat{f}(n) = \begin{cases} \dfrac{(-1)^n e^{-in\pi}}{2n^2} & ; \ n \neq 0 \\ \dfrac{\pi^2}{12} & ; \ n = 0 \end{cases} \implies \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$$

$$\lim_{N \to \infty} S_N(f)(\theta) = f(\theta)$$

$$\frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2} = f(\theta) \implies$$

$\theta = 0$

$$\frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4} \qquad \frac{\pi^2}{4} - \frac{\pi^2}{12}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$