Class 5: Convolutions

Theorem: Let f be a continuous function on the circle. Suppose that

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty.$$

Then

$$S_N(f)(\theta) \longrightarrow f(\theta)$$

uniformly on $\theta \in [-\pi,\pi]$

APPLICATION:

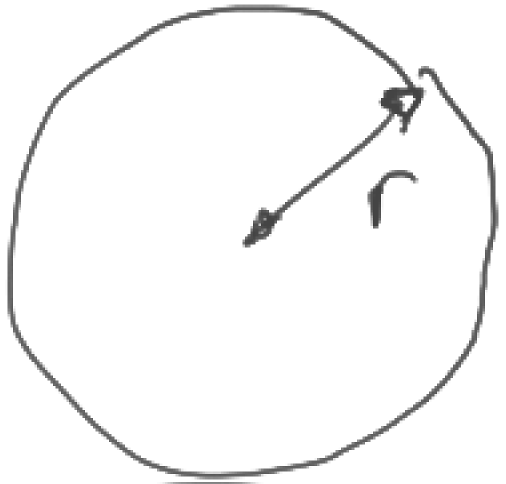
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

? ? ? ?

$$= \frac{\pi^2}{6} \longrightarrow \pi \,?$$



$$S = \pi \cdot r^2$$

?

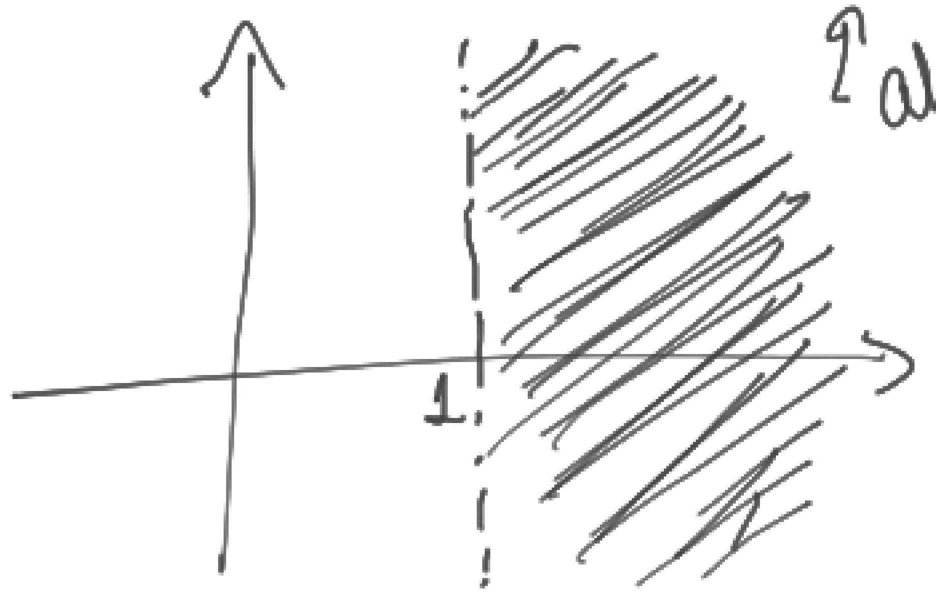
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \,?$$

$\longrightarrow$? f ?

RIEMANN ZETA-FUNCTION

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

absolutely convergent in $\{\mathrm{Re}\, s > 1\}$



$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| < \infty; \quad s \in \mathbb{C}: \ \mathrm{Re}\, s > 1$$

$\zeta(2) = \frac{\pi^2}{6}$; $\zeta(4) = ?$

$\zeta(6) = ?$

$\zeta(2n) = ?$, $n \geq 1$, $n \in \mathbb{Z}$

$\zeta(3)$, $\zeta(5)$, $\zeta(7)$, ...

$\zeta(3) \approx 1.20205\ldots$

$\zeta(5) \approx \ldots.$

$\zeta(s)$ meromorphic function in $\mathbb{C}$

$s = 1$ pole

$\zeta(s) = \sum \frac{1}{n^s}$ ✗

$\zeta(s) := \{ \ldots$

$\mathrm{Re}\, s < 1$

$\zeta(-1) = -\frac{1}{12}$

$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$, $\operatorname{Re} s > 1$

$\zeta(-1) = \sum_{n \geq 1} \frac{1}{n^{-1}}$

$\sum_{n \geq 1} n$

$1+2+3+4+\dots$

$1+2+3+\dots = -\frac{1}{12}$?

Corollary: Let $f : \mathbb{R} \to \mathbb{C}$ such that 2π-periodic and twice continuously differentiable.

Then, $\exists\, M > 0$ such that

$|\hat{f}(n)| \leq \frac{M}{|n|^2}$; $\forall n \in \mathbb{Z} \setminus \{0\}$

$\sum_{n=1}^{N} |\hat{f}(n)| \leq M \sum_{n=1}^{N} \frac{1}{|n|^2} \leq M \cdot \frac{\pi^2}{6}$, $N \geq 1$

$\sum_{n \geq 1}^{\infty} |\hat{f}(n)| < \infty$; $\sum_{n=-\infty}^{-1} |\hat{f}(n)| < \infty$

$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |\hat{f}(n)| < \infty \Rightarrow \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$

$|\hat{f}(0)|$

PROOF: $n \in \mathbb{Z} \setminus \{0\}$,

$f : [0, 2\pi] \to \mathbb{C}$; $\boxed{f(0) = f(2\pi)}$

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cdot \left(\frac{e^{-in\theta}}{-in} \right)' d\theta$$

$$= \frac{1}{2\pi} \left\{ f(\theta) \cdot \frac{e^{-in\theta}}{-in} \Bigg|_{\theta=0}^{\theta=2\pi} - \int_0^{2\pi} f'(\theta) \left(\frac{e^{-in\theta}}{-in} \right) d\theta \right\}$$

$$= \frac{1}{2\pi} \left\{ f(2\pi) \frac{e^{-2\pi i n}}{-in} - f(0) \frac{e^{0}}{-in} + \frac{1}{in} \int_0^{2\pi} f'(\theta) e^{-in\theta} \, d\theta \right\}$$

$$= \frac{1}{2\pi} \left\{ \cancel{\frac{f(2\pi)}{-in}} - \cancel{\frac{f(2\pi)}{-in}} + \frac{1}{in} \int_0^{2\pi} f'(\theta) e^{-in\theta} \, d\theta \right\}$$

$$\hat{f}(n) = \frac{1}{2\pi i n} \int_0^{2\pi} f'(\theta) e^{-in\theta} \, d\theta$$

$$in \hat{f}(n) = \underbrace{\frac{1}{2\pi} \int_0^{2\pi} f'(\theta) e^{-in\theta} \, d\theta}$$

$$\boxed{in \hat{f}(n) = \widehat{f'}(n)} \quad n \neq 0$$

f is continuously differentiable

$f' : [0, 2\pi] \to \mathbb{C}$

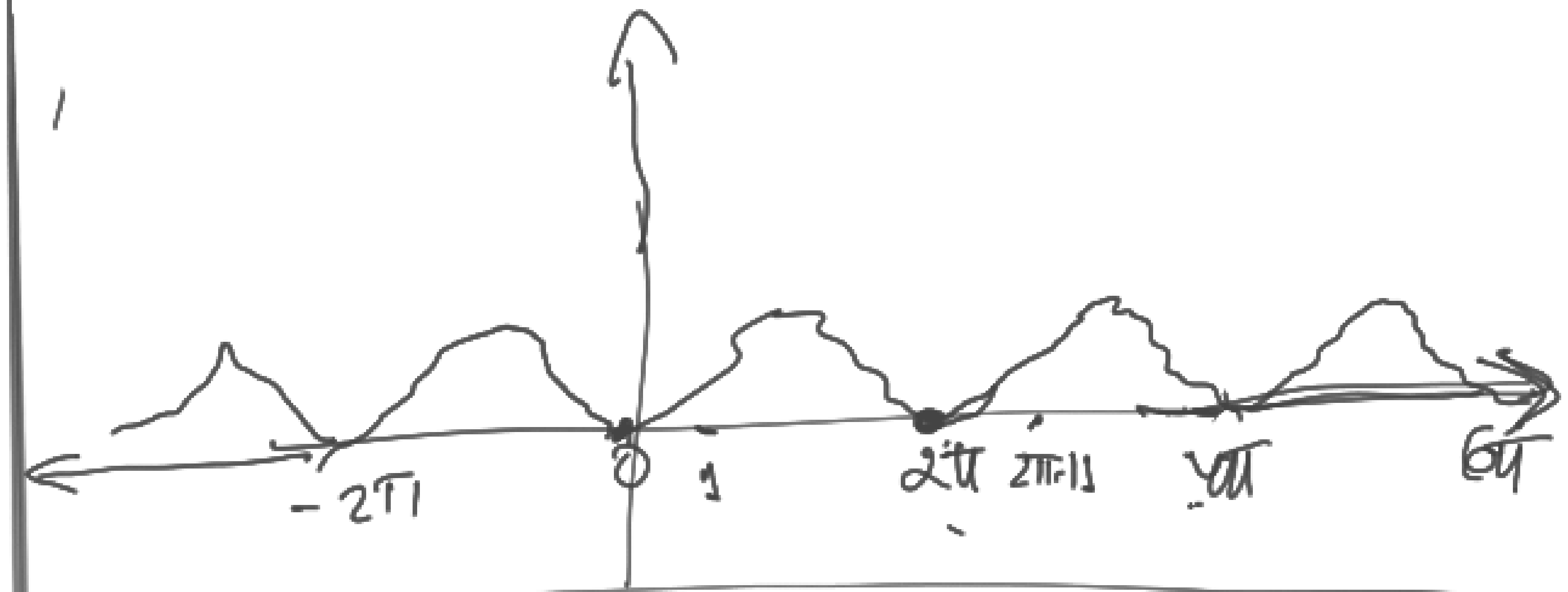
$$\hat{f}(n) = \frac{1}{2\pi i n}\int_0^{2\pi} f'(\theta)\, e^{-in\theta}\, d\theta$$

$$\hat{f}(n) = \frac{1}{2\pi i n}\left[\int_0^{2\pi} f'(\theta)\cdot\left(\frac{e^{-in\theta}}{-in}\right)' d\theta\right]$$

$$= \frac{1}{2\pi i n}\left\{ f'(\theta)\frac{e^{-in\theta}}{-in}\Bigg|_0^{2\pi} - \int_0^{2\pi} f''(\theta)\cdot\left(\frac{e^{-in\theta}}{-in}\right) d\theta\right\}$$

$$= \frac{1}{2\pi i n}\left\{\underbrace{\cancel{\frac{f'(2\pi)}{-in}} - \cancel{\frac{f'(0)}{-in}}} + \frac{1}{in}\int_0^{2\pi} f''(\theta)\, e^{-in\theta}\, d\theta\right\}$$

$$= \frac{1}{2\pi i n \cdot i n}\int_0^{2\pi} f''(\theta)\, e^{-in\theta}\, d\theta$$



$$\boxed{f : \mathbb{R} \to \mathbb{C}\ ;\quad f(x) = f(x+2\pi)}$$

$$\Downarrow$$

$$f'(x) = f'(x+2\pi)$$

$$f'(0) = f'(2\pi)$$

$$\hat{f}(n) = \frac{-1}{2\pi n^2} \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta$$

$$|\hat{f}(n)| = \frac{1}{2\pi n^2} \left| \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta \right|$$

$$\leq \frac{1}{2\pi n^2} \int_0^{2\pi} |f''(\theta) e^{-in\theta}| d\theta$$

$$|\hat{f}(n)| \leq \frac{1}{2\pi n^2} \int_0^{2\pi} |f''(\theta)| d\theta$$

$\theta \in [0, 2\pi]$

$|f''(\theta)| \leq M$; for some $M > 0$.

$$|\hat{f}(n)| \leq \frac{1}{2\pi n^2} \int_0^{2\pi} M \, d\theta$$

$$|\hat{f}(n)| \leq \frac{M}{|n|^2}$$

$$\Rightarrow \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$$

$$\Rightarrow S_N(f) \to f$$

$f : [-\pi, \pi) \to \mathbb{C}$ ✓

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-in\theta} d\theta$$

$$\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$$

$$\cancel{\widehat{fg}(n) = ? \simeq \hat{f}(n)\,\hat{g}(n)?}$$

Let $f, g : \mathbb{R} \to \mathbb{C}$ 2π-periodic integrable on any finite interval.

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$$

; $x \in [-\pi, \pi]$

$g = 1$, $g : \mathbb{R} \to \mathbb{C}$
$g(x) = 1$

$$(f * 1)(x) = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) dy}_{\text{cte}}, \quad x \in [-\pi, \pi)$$

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

$$= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \right) e^{inx}$$

$$= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(y) e^{in(x-y)} dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{in(x-y)} \right) dy$$

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

$$S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) D_N(x-y)\,dy$$

$$S_N(f)(x) = (f * D_N)(x)$$

$$\lim_{N\to\infty} (f * D_N)(x) \ ?$$

PROPOSITION:

Let $f, g, h: \mathbb{R} \to \mathbb{C}$ be 2π-periodic functions and integrable on any finite interval.

i) $f * g = g * f$ ✗

ii) $f*(g+h) = f*g + f*h$ ✗

iii) $f*(g*h) = (f*g)*h$ ✓

iv) $(cf)*g = f*(cg) = c(f*g)$, $\forall c \in \mathbb{C}$

v) $f*g$ is continuous ✓

vi) $\widehat{f*g}(n) = \hat{f}(n)\cdot\hat{g}(n)$ ✓

$\forall n \in \mathbb{Z}$

PROOF:

i) $(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$

$x-y = z$

$= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-y).g(y)\, dy$

$(g*f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\, f(x-y)\, dy$

iii) f, g are continuous in $\mathbb{R}$.

$\int_a^b \left(\int_a^b f(x-y)\, dx\right) g(y)\, dy$

$= \int_a^b \left(\int f \cdot g\, dy\right) dx$

Lemma: Let f be a function on the circle. Suppose that f is integrable and $|f(x)| \leq B$, $\forall x \in [-\pi, \pi]$

$\Rightarrow$ There exist a sequence $\{f_K\}_{K\geq 1}$ of functions on the circle such that

i) f_K are continuous

ii) $|f_K(x)| \leq B$ $\forall K \geq 1$; $\forall x \in [-\pi, \pi]$

iii) $\lim_{K\to\infty} \int_{-\pi}^{\pi} |f_K(x) - f(x)|\, dx = 0$