Class 10: VECTOR SPACES AND FOURIER SERIES

THURSDAY 04: ZOOM

TUESDAY: 09: LECTURE IN F2

- ALIVE BY ZOOM.
- NOT RECORDED.

THURSDAY 11 DEPENDENCE ON TUESDAY 09

Let f be a function on the circle (integrable)

$$S_N(f)(\theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta} \quad \text{"}(|n| \leq N)\text{"}$$

$$¿\, S_N(f)(\theta) \longrightarrow f(\theta)\,?$$

$f \in C^2([-\pi,\pi]) \Rightarrow S_N(f)(\theta) \to f(\theta)$ uniformly

For each θ: $\lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$

$$\boxed{f \in C^1([-\pi,\pi]) \Rightarrow \lim_{N\to\infty} S_N(f)(\theta) = f(\theta)}$$

$$\boxed{\lim_{N\to\infty} \int_0^{2\pi} |S_N(f)(\theta) - f(\theta)|^2 \, d\theta}$$

VECTOR SPACES (on $\mathbb{R}$ or $\mathbb{C}$)

$x, y \in \mathbb{V}$, $\boxed{\lambda \in \mathbb{R} \text{ (or } \mathbb{C})} \Rightarrow \mathbb{F}$

- $x + y \in \mathbb{V}$, $\lambda x \in \mathbb{V}$

$\lambda, \mu \in \mathbb{F}$

- $x + y = y + x$
- $(x+y)+z = x+(y+z)$
- $\exists\, e \in V$: $x + e = x$
- $\forall x \in V \ \exists\, -x \in V$ $\quad x + (-x) = e$

- $(\lambda\mu)x = \lambda(\mu x) = \mu(\lambda x)$
- $(\lambda+\mu)x = \cancel{\lambda x} + \mu x$
- $\vdots$

$\mathbb{R}^n$; $(x_1, x_2, \ldots x_n)$ $\quad x_i \in \mathbb{R}$

$\mathbb{C}^n$; $(\omega_1, \omega_2, \ldots, \omega_n)$ $\quad \omega_i \in \mathbb{C}$

INNER PRODUCT: $x, y \in V$

$\langle x, y \rangle \in \mathbb{F}$

1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (when $\mathbb{F} = \mathbb{C}$) - HERMITIAN

$\langle x, y \rangle = \langle y, x \rangle$ (when $\mathbb{F} = \mathbb{R}$)

$\alpha, \beta \in \mathbb{F}$

2) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

3) $\langle x, x \rangle \geqslant 0 \quad \forall x \in V$

POSITIVE - DEFINITE

*) $\langle x, x \rangle = 0 \Longrightarrow x = 0.$

STRICTLY POSITIVE - DEFINITE

NORM: $\boxed{\|x\| = \langle x, x \rangle^{1/2}}$

PROPERTIES OF THE NORM:

i) PYTHAGOREAN THEOREM: $x, y \in V$

$\boxed{\langle x, y \rangle = 0}$ (x and y are orthogonal)

$$\Rightarrow \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

ii) CAUCHY - SCHWARZ INEQUALITY

$x, y \in V$

$$|\langle x, y \rangle| \leq \|x\| . \|y\|$$

iii) TRIANGLE INEQUALITY:

$$\|x+y\| \leq \|x\| + \|y\|.$$

PROOF:

i) $\|x+y\|^2 = \langle x+y, x+y \rangle$

$$= \langle x, x \rangle + \underbrace{\langle x, y \rangle}_{0} + \underbrace{\langle y, x \rangle}_{0} + \langle y, y \rangle$$

$$= \|x\|^2 + \|y\|^2$$

ii) *) $\|y\| = 0$ Let ($c \in \mathbb{R}$)

$\langle x, cy \rangle = \bar{c}\langle x, y \rangle$

$0 \leq \|x + cy\|^2 = \langle x+cy, x + cy \rangle$

$$= \langle x, x \rangle + \langle x, cy \rangle + \langle cy, x \rangle + \langle cy, cy \rangle$$

$$= \|x\|^2 + \bar{c}\langle x, y \rangle + c\langle y, x \rangle + c.\bar{c}\underbrace{\langle y, y \rangle}_{0}$$

$c \in \mathbb{R}$

$$= \|x\|^2 + \bar{c}\langle x, y \rangle + c. \overline{\langle x, y \rangle}$$

$$= \|x\|^2 + c\langle x, y \rangle + c\,\overline{\langle x, y \rangle}$$

$$= \|x\|^2 + c.2.\mathrm{Re}\langle x, y \rangle$$

*) $\|y\| = 0$: For any $c \in \mathbb{R}$

$$0 \le \|x\|^2 + 2c \, \mathrm{Re}\{\langle x, y\rangle\}$$

if $\mathrm{Re}\langle x,y\rangle > 0$ $(c \to -\infty)$ CONTRAD.

if $\mathrm{Re}\langle x,y\rangle < 0$ $(c \to +\infty)$ CONTRAD

$\Rightarrow \mathrm{Re}\langle x,y\rangle = 0$

$c \in \mathbb{R}$

$$0 \le \|x + icy\|^2 = \dots$$

$\Rightarrow \mathrm{Im}\langle x,y\rangle = 0$

Therefore; if $\|y\| = 0 \Rightarrow \langle x,y\rangle = 0$

$$|\langle x,y\rangle| \le \|x\| \cdot \|y\|$$

$0 \qquad 0$

*) if $\|y\| > 0$: Take $c = \dfrac{\langle x,y\rangle}{\langle y,y\rangle}$

$$\langle x - cy, cy\rangle = \bar{c}\,\langle x - cy, y\rangle = \bar{c}\left(\langle x,y\rangle - \langle cy,y\rangle\right)$$
$$= \bar{c}\left(\langle x,y\rangle - c\langle y,y\rangle\right) = 0$$

$$\langle x - cy, cy\rangle = 0$$

$$\Rightarrow \|x\|^2 = \|x - cy\|^2 + \|cy\|^2$$

$$\ge \|cy\|^2 = |c|^2 \cdot \|y\|^2$$

$$\Rightarrow \|x\| \ge |c| \cdot \|y\| = \left|\frac{\langle x,y\rangle}{\langle y,y\rangle}\right| \cdot \|y\|$$

$$= \frac{|\langle x,y\rangle|}{\|y\|^2} \cdot \|y\|$$

$$= \frac{|\langle x,y\rangle|}{\|y\|} \Longrightarrow \|x\| \cdot \|y\| \ge |\langle x,y\rangle|$$

$$\|x\| \cdot \|y\| \ge |\langle x,y\rangle|$$

when

$$\|x\| \cdot \|y\| = |\langle x,y\rangle|$$

$$\|x - cy\|^2 = 0$$

$$\|x - cy\| = 0$$

INNER PRODUCT IS STRICTLY POSITIVE DEFINITE

$$\boxed{x = cy}$$

$$3 + 8c \ge 0 \quad \forall c \in \mathbb{R}$$

$$c = -50000$$

iii) $\|x+y\| \le \|x\| + \|y\|$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$
$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$
$$= \|x\|^2 + 2\underbrace{\mathrm{Re}\langle x, y\rangle} + \|y\|^2$$
$$\le \|x\|^2 + 2\underbrace{|\langle x, y\rangle|} + \|y\|^2$$
$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2$$

$$\|x+y\| \le \|x\| + \|y\|$$

$\mathbb{R}^n$, $\mathbb{C}^n$

$x \in \mathbb{R}^n$; $x = (x_1, \dots x_n) \Rightarrow \|x\| = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$

$y = (y_1, \dots y_n)$ $\quad \langle x, y\rangle = x_1 y_1 + \dots + x_n y_n$

$\mathbb{C}^n$

$$\ell_2(\mathbb{Z}) = \left\{ (\dots, a_{-1}, a_0, a_1, \dots);\ a_i \in \mathbb{C} \text{ and } \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty \right\}$$

$(, \dots a_{-1}, a_0, a_1, \dots)$; $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$

$(, \dots b_{-1}, b_0, b_1, \dots)$; $\sum_{n\in\mathbb{Z}} |b_n|^2 < \infty$

$\Downarrow$

$(, \dots, \underbrace{a_{-1}+b_{-1}}_{c_{-1}}, \underbrace{a_0+b_0}_{c_0}, \underbrace{a_1+b_1}_{c_1}, \dots)$; $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$

$a,b\geq 0$ $(a+b)^2 \leq 2(a^2+b^2)$

$$|c_n|^2 = |a_n+b_n|^2 \leq (|a_n|+|b_n|)^2$$

$$\leq 2(|a_n|^2+|b_n|^2)$$

$$\sum_{n=-M}^{N} |c_n|^2 \leq 2\underbrace{\sum_{n=-M}^{N}|a_n|^2}_{\downarrow} + 2\sum_{n=-M}^{N}|b_n|^2$$

$$\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$$

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\ldots \stackrel{\Rightarrow}{=} \sum_{n=-N}^{N} \hat{f}(n)\ldots$$ FOURIER SERIES

$a_n \geq 0$; $0 \leq \sum_{n\in\mathbb{Z}} a_n < \infty$?

$$\sum_{n=-M}^{N} a_n \leq \sum_{n=-N}^{N} a_n < \infty$$

$N \geq M$

$$\sum_{n=-N}^{N} a_n$$

$\sum_{n\in\mathbb{Z}} b_n$ is absolutely convergent

$$\sum_{n\in\mathbb{Z}} |b_n| < \infty \longrightarrow \sum_{n=-N}^{N} |b_n|\ ?$$

$a, b \in \ell^2(\mathbb{Z})$ $\quad a = (\ldots\ldots, a_0, \ldots)$

$\quad b = (\ldots\ldots, b_0, \ldots)$

$$\boxed{\langle a, b\rangle = \sum_{n\in\mathbb{Z}} a_n \overline{b_n}}$$

$\to$ INNER PRODUCT

$\downarrow$

ABSOLUTELY CONVERGENT

EXERCISE: PROVE THAT $\ell^2(\mathbb{Z})$ IS A VECTOR SPACE WITH INNER PRODUCT.

Let f, g be integrable functions on the circle.

VECTOR SPACE: {INTEGRABLE FUNCTIONS ON THE CIRCLE}

$\parallel$

V

Let $f, g \in V$

$$\boxed{\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\overline{g(\theta)}\,d\theta} \checkmark$$

$$\langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\overline{f(\theta)}\,d\theta$$

$$\boxed{\langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\,d\theta} \geq 0$$

$$\|f\| = \langle f, f\rangle^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\,d\theta\right)^{1/2}$$

Let $n \in \mathbb{Z}$: $\quad e_n(\theta) = e^{in\theta}$

ORTHONORMAL

$$\{e_n\}_{n\in\mathbb{Z}} \Rightarrow \langle e_n, e_m\rangle = \begin{cases} 1; & m = n \\ 0 & m \neq n \end{cases}$$

$$\langle e_n, e_m\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} \cdot e^{-im\theta}\,d\theta$$

$n \in \mathbb{Z}$

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, \overline{e^{in\theta}}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \cdot \overline{e_n(\theta)}\, d\theta$$

$$\boxed{\hat{f}(n) = \langle f; e_n \rangle}$$

$$S_N(f)(\theta) = \sum_{|n|\le N} \hat{f}(n) e^{in\theta} = \sum_{|n|\le N} \langle f, e_n \rangle e_n(\theta)$$

$$\boxed{S_N(f) = \sum_{|n|\le N} \langle f, e_n \rangle e_n}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |S_N(f) - f(\theta)|^2 d\theta \; ? \xrightarrow{N \to \infty} 0$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (S_N(f) - f(\theta))\overline{(S_N(f) - f(\theta))}\, d\theta = \|S_N(f) - f\|^2$$

Let $|m| \le N$

$$\langle f - S_N(f); e_m \rangle = \langle f, e_m \rangle - \langle S_N(f); e_m \rangle$$

$$= \langle f, e_m \rangle - \left\langle \sum_{|n|\le N} \langle f, e_n \rangle e_n ; e_m \right\rangle$$

$$= \langle f, e_m \rangle - \sum_{|n|\le N} \langle f, e_n \rangle \cdot \underbrace{\langle e_n, e_m \rangle}_{\begin{cases} 0 & m \ne n \\ 1 & m = n \end{cases}}$$

$$= \langle f, e_m \rangle - \langle f; e_m \rangle \cdot 1$$

$$= \langle f, e_m \rangle - \langle f, e_m \rangle = 0$$

$\therefore$ $$\boxed{|m| \le N \quad \langle f - S_N(f), e_m \rangle = 0}$$

$$\Rightarrow \langle f - S_N(f); \sum_{|m|\le N} a_m e_m \rangle = 0 \quad \forall \{a_m\} \in \mathbb{C}$$

TRIG. POLYNOMIAL

$$\langle f - S_N(f) ; \sum_{|n|\leq N} a_n e_n \rangle = 0$$

for any $\{a_n\}_{|n|\leq N} \subset \mathbb{C}$

$$\boxed{\langle f - S_N(f) ; S_N(f) \rangle = 0}$$

$$\boxed{\|f\|^2 = \|f - S_N(f)\|^2 + \|S_N(f)\|^2}$$

PITAGORAS THEOREM

$$\|S_N(f)\|^2 = \langle S_N(f) ; S_N(f) \rangle$$

$$= \langle \sum_{|n|\leq N} \langle f, e_n\rangle e_n ; \sum_{|m|\leq N} \langle f, e_m\rangle e_m \rangle$$

$$= \sum_{|n|\leq N} \langle f; e_n\rangle \cdot \overline{\langle f, e_n\rangle} \cdot \langle e_n, e_n\rangle$$

$$= \sum_{|n|\leq N} |\langle f; e_n\rangle|^2 \cdot 1 = \sum_{|n|\leq N} |\langle f; e_n\rangle|^2$$

$$= \sum_{|n|\leq N} |\hat{f}(n)|^2$$

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{|n|\leq N} |\hat{f}(n)|^2$$