(04.02.21)

Summary:

No webcam. :-($-24°C$ in Trondheim.!

Last bits of Fourier series

Then Fourier Transform.

Next tuesday: Physical Class 09.02. in F2. Will use Mac on Zoom. same zoom link.

Written Exam!

CLASS 11

Vector spaces

V is a complex vector space.

$\langle \cdot , \cdot \rangle$ i.p. (inner product) → $\langle x,y \rangle = \overline{\langle y,x \rangle}$ → linear → $\langle x,x \rangle \geq 0$ pos. definite.

$\|x\| = \langle x, x \rangle^{1/2}$ (norm)

*) $\langle x,y \rangle = 0$: $x \perp y$ $\Rightarrow$ $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

*) $|\langle x,y \rangle| \leq \|x\| \cdot \|y\|$

*) $\|x+y\| \leq \|x\| + \|y\|$

$V = \mathbb{C}$; $\mathbb{R}$; $\mathbb{C}^d$; $\mathbb{R}^d$

basis : $(1,0,\dots,0)$
$(0,1,\dots,0)$
$\vdots$
$(0,\dots,0,1)$

$$V = \ell^2(\mathbb{Z}) = \{ (\dots, a_{-1}, a_0, a_1, \dots) ; a_j \in \mathbb{C} \ \forall j \in \mathbb{Z} : \sum_{j\in\mathbb{Z}} |a_j|^2 < \infty . \}$$

$\ell^2(\mathbb{Z})$ is a vector space. [Ex 3: prove this!]

Let $A, B \in \ell^2(\mathbb{Z})$

$$A = (\dots, a_{-1}, a_0, a_1 \dots)$$
$$B = (\dots, b_{-1}, b_0, b_1, \dots)$$

$$\langle A, B \rangle = \underbrace{\sum_{n\in\mathbb{Z}} a_n \overline{b_n}}_{} \quad \in \mathbb{C} \quad \leftarrow \text{This guy exists!}$$

$$\text{if } A, B \in \ell^2(\mathbb{Z}) \Rightarrow \sum_{n\in\mathbb{Z}} |a_n \overline{b_n}| < \infty \quad \text{abs. convers.}$$

- $\langle A, B \rangle = \overline{\langle B, A \rangle}$
- Linear
- $\langle A, A \rangle \geq 0 \Leftrightarrow \sum_{n\in\mathbb{Z}} a_n \overline{a_n} \geq 0$

$$\Leftrightarrow \sum_{n\in\mathbb{Z}} |a_n|^2 \geq 0 .$$

$$\|A\| = \langle A, A \rangle^{1/2} = \Big(\sum_{n\in\mathbb{Z}} a_n \overline{a_n} \Big)^{1/2} = \Big(\sum_{n\in\mathbb{Z}} |a_n|^2 \Big)^{1/2}$$

Hilbert space: Two additional properties.

i) $\|x\| = 0 \Rightarrow x = 0$ (STRICTLY POS DEFINITE)

ii) V is a complete space → Every Cauchy sequence has a limit L in the vec. space. ($L \in V$.)

if only i) or ii) but not both: pre-Hilbert space if both i) and ii) works:

$\Rightarrow$ V is a Hilbert space.

Cauchy Sequence: $\{a_n\}_{n\in\mathbb{Z}}$ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : |m|, |n| > N$

$$\Rightarrow \|a_m - a_n\| < \varepsilon$$

Norm gives you a notion of distance.

$\ell^2(\mathbb{Z})$ is a Hilbert space:

1) $\|x\| = 0 \Rightarrow \sum_{n\in\mathbb{Z}} |a_n|^2 = 0 \Rightarrow 0 \leq |a_n| \leq \sum_{n\in\mathbb{Z}} |a_n|^2 = 0$

$$|a_n| = 0 \Rightarrow a_n = 0 \ \forall n \in \mathbb{Z}$$

$$\Rightarrow A = \underline{0}$$

2) $\{A_n\}$ is a Cauchy sequence has a limit in $\ell^2(\mathbb{Z})$.

Exercise!

Let R be a family of integrable functions on $[0, 2\pi]$.

$$f : [0, 2\pi] \to \mathbb{C}.$$

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta$$

$$\langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\, d\theta \geq 0.$$

$$\|f\| = \langle f, f\rangle^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\, d\theta\right)^{1/2}$$

C-S: $|\langle f, g\rangle|^2 \leq \|f\|^2 \|g\|^2$

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta \right|^2 \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta \right) \left(\frac{1}{2\pi} \int_0^{2\pi} |g(\theta)|^2 d\theta \right)$$

Powerful inequality!

But we can prove something more:

$$A, B \in \mathbb{R} \quad : \quad 2AB \leq A^2 + B^2 \qquad \left((A-B)^2 \geq 0 \right)$$

Let $\lambda > 0$: $A := \lambda^{1/2} |f(\theta)|$, $B := \lambda^{-1/2} |g(\theta)|$

$$2\lambda^{1/2} |f(\theta)|\, \lambda^{-1/2} |g(\theta)| \leq \lambda |f(\theta)|^2 + \lambda^{-1} |g(\theta)|^2$$

$$\frac{2}{2\pi} \int_0^{2\pi} |f(\theta)||g(\theta)|\, d\theta \leq \frac{\lambda}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta + \frac{\lambda^{-1}}{2\pi} \int_0^{2\pi} |g(\theta)|^2 d\theta$$

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)||g(\theta)|\, d\theta \leq \frac{\lambda \|f\|^2}{2} + \frac{\lambda^{-1}}{2} \|g\|^2$$

Pick now: $\lambda = \|g\| / \|f\|$ $\quad (\|g\| \neq 0 ; \|f\| \neq 0)$

$$\Longrightarrow \qquad \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)||g(\theta)|\, d\theta \leq \|g\| \cdot \|f\|$$

But
$$|\langle f, g \rangle| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \overline{g(\theta)}\, d\theta \right|$$
$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)||g(\theta)|\, d\theta$$
$$\leq \|f\| \cdot \|g\|$$

Better than C-S in this space.

Property one?

1) $\|f\| = 0 \Rightarrow f = 0$?

$$\|f\| = 0 \iff \boxed{\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 \, d\theta = 0} \Rightarrow f = 0 \, ?$$

$$g(\theta) = \begin{cases} 0 & ; \ 0 \le \theta < 2\pi \\ 1 & ; \ \theta = 2\pi . \end{cases}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |g(\theta)|^2 \, d\theta = 0 \qquad \text{so: } \|g\| = 0$$

but $g \neq 0$.

Property one doesn't work! ($\mathcal{R}$ is not a Hilbert space.)

$\mathcal{R}$: $f = 0$ iff $\iff$ $f = 0$ in all pts. of continuity.

$$g(\theta) = \begin{cases} 0 & ; \ 0 \le \theta \le 2\pi \\ 2 & ; \ \theta = 2\pi \end{cases} \longrightarrow g \equiv 0.$$

$$h(\theta) = \begin{cases} 0 & ; \ 0 < \theta \le 2\pi \\ 1 & ; \ \theta = 0 \\ 2 & ; \ \theta = 2\pi \end{cases} \longrightarrow h \equiv 0.$$

$$\boxed{\text{Points of discontinuity has measure zero.}}$$

has measure 0 b/c it's countable

$$F(\theta) = \begin{cases} 1 & ; \ \theta = \overbrace{\{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \tfrac{1}{n}\}} \\ 0 & ; \ \theta \in (0, 2\pi) \setminus \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{n} \dots\} \end{cases}$$

$$\longrightarrow F \equiv 0.$$

Property 2 of Hilbert doesn't work for $\mathcal{R}$. EXERCISE!

$$\boxed{\begin{array}{c}\{f_n\} \text{ Cauchy seqnce } f_n \in \mathcal{R} \\ \|f_n - f\| \to 0 \quad (f \notin \mathcal{R})\end{array}}$$

$\mathcal{R}$: IS NOT A HILBERT SPACE!

BREAK!

Exam: if you solve all exercises, exam should be fine!

Book is not so strong!

$\mathcal{R}$: family of integrable functions on circle. (A pre-Hilbert space)

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta$$

$$\| f\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta\right)^{1/2}$$

$e_n(\theta) = e^{in\theta}$; $n \in \mathbb{Z}$ $\Rightarrow$ $\{e_n\}_{n\in\mathbb{Z}}$ ORTHONORMAL

$$\langle e_n, e_m\rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m.\end{cases}$$

and $\boxed{\langle f, e_n\rangle = \hat{f}(n) \quad \forall n \in \mathbb{Z}}$

$$S_N(f)(\theta) = \sum_{|n|\leq N} \hat{f}(n) e^{in\theta} = \sum_{|n|\leq N} \langle f, e_n\rangle e_n(\theta)$$

$$\boxed{S_N(f) = \sum_{|n|\le N} \langle f, e_n\rangle e_n}$$

$|n|\le N$:
$$\langle f - S_N(f), e_n\rangle = \langle f, e_n\rangle - \langle S_N(f), e_n\rangle$$
$$= \langle f, e_n\rangle - \langle f, e_n\rangle \underbrace{\langle e_n, e_n\rangle}_{1}$$
$$= 0$$

So: $\langle f - S_N(f), e_n\rangle = 0 \quad \forall\, |n|\le N$

$$\Downarrow$$

$$\langle f - S_N(f), \sum_{|n|\le N} b_n e_n\rangle = 0 \quad \forall\, b_n \in \mathbb{C}.$$

$f - S_N(f) \perp$ all trig. polynomials of degree less than or equal to N.

$$\langle f - S_N(f), \sum_{|n|\le N} b_n e_n\rangle = 0 \quad \forall\, b_n \in \mathbb{C}$$

in particular
Let now $b_n = \hat{f}(n)$:

$$\langle f - S_N(f), S_N(f)\rangle = 0$$

Pythagoras:
$$\|f\|^2 = \|f - S_N(f)\|^2 + \|S_N(f)\|^2$$

$$\|S_N(f)\|^2 = \langle S_N(f), S_N(f)\rangle = \langle \sum_{|n|\le N} \hat{f}(n) e_n, \sum_{|m|\le N} \hat{f}(m) e_m\rangle$$

$$= \sum_{|n| \leq N} \hat{f}(n) \overline{\hat{f}(n)} \langle e_n, e_n \rangle$$

$$= \sum_{|n| \leq N} |\hat{f}(n)|^2$$

So: $$\boxed{\| S_N(f) \|^2 = \sum_{|n| \leq N} |\hat{f}(n)|^2}$$

Lemma: **BEST APPROXIMATION**

Let f be an integrable function on the circle.

Then $\| f - S_N(f) \| \leq \| f - \sum_{|n| \leq N} c_n e_n \|$ for any $c_n \in \mathbb{C}$.

Equality holds when $c_n = \hat{f}(n)$.

Proof: $c_n \in \mathbb{C}$:

$$\| f - \sum_{|n| \leq N} c_n e_n \| = \| f - S_N(f) + S_N(f) - \sum_{|n| \leq N} c_n e_n \|$$

$$= \| \underbrace{f - S_N(f)}_{A} + \underbrace{\sum_{|n| \leq N} (\hat{f}(n) - c_n) e_n}_{B} \|$$

$A \perp B$ by above argument.

Then: $$\| f - \sum_{|n| \leq N} c_n e_n \|^2 \overset{\text{Pythagoras}}{=} \| f - S_N(f) \|^2 + \underbrace{\| \sum_{|n| \leq N} (\hat{f}(n) - c_n) e_n \|^2}_{\geq 0}$$

$$\geq \| f - S_N(f) \|^2$$

$$\Big\| f - \sum_{|n|\le N} c_n e_n \Big\| \geq \| f - S_N(f) \|$$

Equality only when $\underbrace{c_n = \hat{f}(n)}$.

$$\Big\| f - \sum_{|n|\le N} c_n e_n \Big\|^2 = \| f - S_N(f) \|^2 + \underbrace{\Big\| \sum_{|n|\le N} (\overbrace{\hat{f}(n) - c_n}^{=0}) e_n \Big\|^2}_{\geq 0}$$

$$\geq \| f - S_N(f) \|^2$$

$$\Big\| \sum_{|n|\le N} (\underbrace{\hat{f}(n) - c_n}_{=: d_n}) e_n \Big\| = 0$$

$$\Rightarrow \Big\| \sum_{|n|\le N} d_n e_n \Big\| = 0 \implies \Big\langle \sum_{|n|\le N} d_n e_n , \sum_{|m|\le N} d_m e_m \Big\rangle = 0$$

$$\Rightarrow \sum_{|n|\le N} |d_n|^2 = 0$$

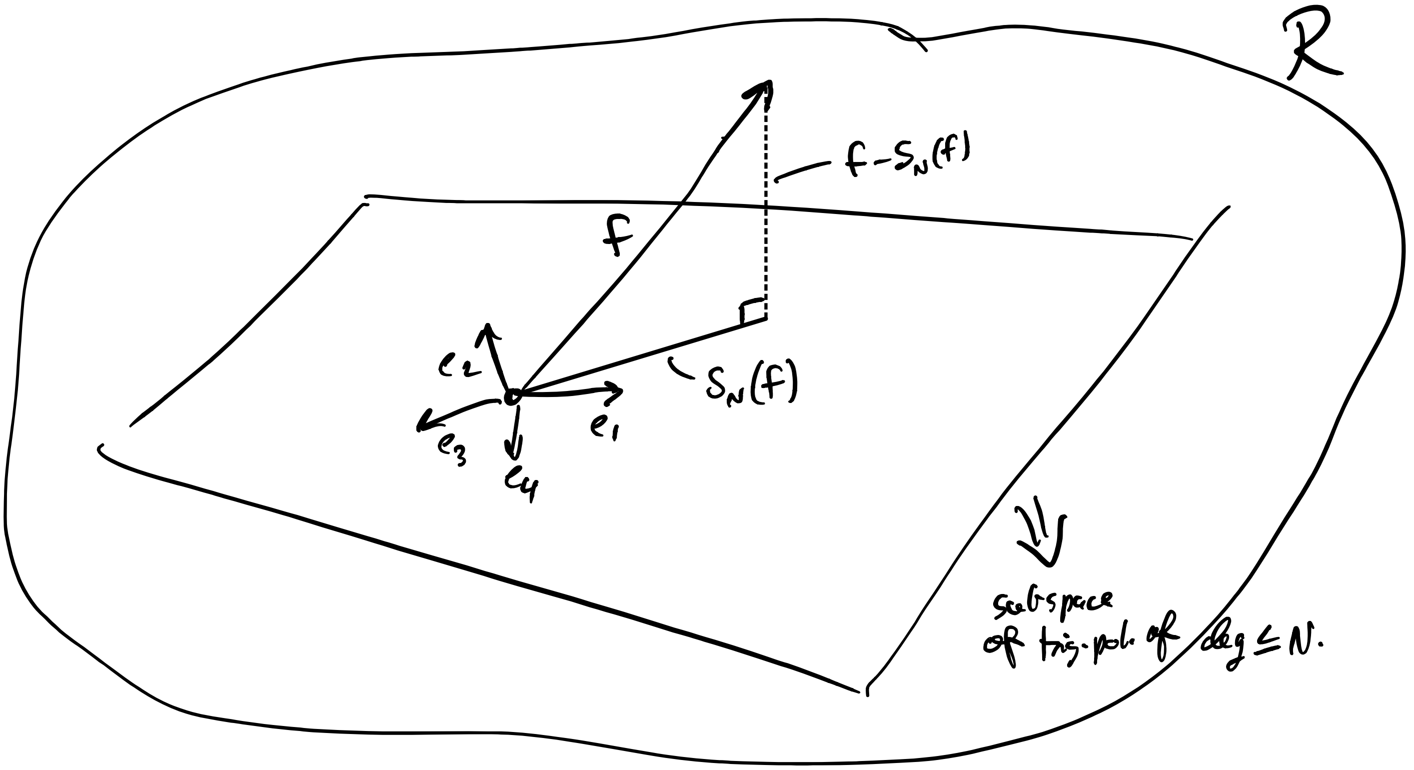
$$\Rightarrow |d_n| = 0 \quad \forall n$$

$$\Rightarrow d_n = 0.$$

$$\Rightarrow \hat{f}(n) - c_n = 0$$

$$\Rightarrow \boxed{c_n = \hat{f}(n) \quad \forall n}$$

$$\Big\| f - \sum_{|n|\le N} c_n e_n \Big\| = \| f - S_N(f) \| \implies c_n = \hat{f}(n).$$



$S_N(f)$ is the projection of f into the trig. polynomials of deg $\leq N$.

" $\|f - S_N(f)\| \longrightarrow 0$ " Next up!

Ex. class in person next time. Physical. Use mask!