Class 16:

Riemann - Lebesgue Lemma V1: If f is integrable on the circle, then

$$\lim_{|n|\to\infty} \hat{f}(n) = 0.$$

Riemann - Lebesgue Lemma V2: If f is integrable on the circle, then

$$\lim_{n\to\infty} \int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta = 0, \text{ and } \lim_{n\to\infty} \int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta = 0.$$

PROOF: $V1 \Leftrightarrow V2$. f integrable on the circle

- $V_2 \to V_1$
- $V_1 \to V_2$

$$\lim_{n\to\infty} \int_0^{2\pi} (\operatorname{Re} f(\theta))\, e^{-in\theta}\,d\theta = 0 \quad \dots (a)$$

$$\lim_{n\to\infty} \int_0^{2\pi} (\operatorname{Im} f(\theta))\, e^{-in\theta}\,d\theta = 0 \quad \dots (b)$$

$$\lim_{n\to\infty} \int_0^{2\pi} (\operatorname{Re} f(\theta))(\cos(n\theta) - i\sin(n\theta))\,d\theta = 0$$

$$\lim_{n\to\infty} \int_0^{2\pi} (\operatorname{Re} f(\theta)\cos(n\theta) - i\operatorname{Re} f(\theta)\sin(n\theta))\,d\theta = 0$$

$$\Rightarrow \lim_{n\to\infty} \int_0^{2\pi} \operatorname{Re} f(\theta)\cos(n\theta)\,d\theta = 0 \quad \dots a.1$$

$$\lim_{n\to\infty} \int_0^{2\pi} \operatorname{Re} f(\theta)\sin(n\theta)\,d\theta = 0 \quad \dots a.2$$

$$\lim_{n\to\infty} \int_0^{2\pi} \operatorname{Im} f(\theta)\cos(n\theta)\,d\theta = 0 \quad \dots b.1$$

$$\lim_{n\to\infty} \int_0^{2\pi} \operatorname{Im} f(\theta)\sin(n\theta)\,d\theta = 0 \quad \dots b.2$$

a.1, b.1) $$\lim_{n\to\infty} \int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta = 0$$

a.2, b.2) $$\lim_{n\to\infty} \int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta = 0$$

$$S_N(f)(\theta_0) - f(\theta_0) = I_1 + I_2$$

Riemann-Lebesgue V.2

$$N\to\infty \quad S_N(f)(\theta_0) \longrightarrow f(\theta_0)$$

Corollary: Let f, g two integrable functions on the circle. Let $\theta_0 \in I \to$ open interval, and suppose that

$$f(\theta) = g(\theta) \quad \forall \theta \in I$$

$$\Rightarrow \lim_{N \to \infty} (S_N f)(\theta_0) - (S_N g)(\theta_0) = 0.$$

PROOF:

$$S_N(f)(\theta) = \sum_{|n| \le N} \hat{f}(n) e^{in\theta}$$

$$h = f - g \Rightarrow \boxed{h(\theta) = 0} \quad \forall \theta \in I$$

h integrable

h is differentiable at θ_0

$$\Rightarrow \underbrace{S_N(h)(\theta_0)}_{(S_N f)(\theta_0) - (S_N g)(\theta_0) \to 0} \to h(\theta_0)$$

APPLICATIONS IN NUMBER THEORY:

$x \in \mathbb{R}$ $\qquad \mathbb{R} = \biguplus_{n \in \mathbb{Z}} [n; n+1)$

$x \in \mathbb{R} \quad \exists\, n \in \mathbb{Z}: \quad n \le x < n+1 \,; \leftarrow n = \underline{[x]}$

$$x = [x] + \underbrace{x - [x]}_{\langle x \rangle}$$

$$[x] \le x < [x]+1$$

$$0 \le \underbrace{x - [x]}_{\langle x \rangle} < 1$$

i) $[x] \in \mathbb{Z}$

ii) $\langle x \rangle \in [0; 1)$

$\langle 5.5 \rangle = 0.5$

$\langle 6.25 \rangle = 0.25$

$\langle \pi \rangle = \ldots$

$\langle 6.5 \rangle = 0.5$

$\langle 7.5 \rangle = 0.5$

$$\langle \cdot \rangle : \mathbb{R} \to [0,1)$$
$$x \mapsto \langle x \rangle$$

$x, y \in \mathbb{R}$

$$\underbrace{x \sim y}_{x \equiv y \text{ MOD } 1} \Longleftrightarrow x - y \in \mathbb{Z}$$

$$x \equiv y \text{ MOD } 1 \Longleftrightarrow x - y \in \mathbb{Z}$$

- $x \equiv x$ MOD 1
- $x \equiv y$ MOD 1 $\Leftrightarrow$ $y \equiv x$ MOD 1
- $x \equiv y$ MOD 1, $y \equiv z$ MOD 1 $\Rightarrow x \equiv z$ MOD 1

CLASS OF EQUIVALENCE OF X

$\bar{x} = \{ y : y \equiv x \bmod 1 \}$

$y \in \bar{x} \Leftrightarrow y \equiv x \bmod 1$

$\Leftrightarrow y - x \in \mathbb{Z}$

$\langle y \rangle = \langle x \rangle$ FRACTIONAL PART

$[\![y]\!] + \langle y \rangle - [\![x]\!] + \langle x \rangle \in \mathbb{Z}$

$\langle y \rangle - \langle x \rangle \in \mathbb{Z}$

$0 \leq \langle y \rangle < 1$ ✓

$-1 < -\langle x \rangle \leq 0$ ✓ $\downarrow (+)$

$-1 < \langle y \rangle - \langle x \rangle < 1$

$\langle y \rangle - \langle x \rangle = 0$

$\boxed{\langle y \rangle = \langle x \rangle}$

EXERCISE

$y : \langle y \rangle = \langle x \rangle$

$y - [\![y]\!] = x - [\![x]\!]$

$y - x = [\![y]\!] - [\![x]\!] \in \mathbb{Z}$

$y \equiv x \bmod 1$

$\bar{x} = \{ y : \langle y \rangle = \langle x \rangle \}$

$\boxed{9:21 \text{ AM}}$

$[\![x]\!] \quad [\![x]\!], \quad [x], \quad [\![x]\!]$ } INTEGER PART

$[0,1)$

$\langle x \rangle = \langle \langle x \rangle \rangle$ $\in [0,1)$

$\cdot)\ y \in \bar{x} \Leftrightarrow \langle y \rangle = \langle x \rangle \Leftrightarrow y \sim \langle x \rangle$

$\cdot)\ \boxed{\cancel{x} \equiv \langle x \rangle \bmod 1}$ unique element in $[0,1)$

Let $\gamma \neq 0$

$\{n\gamma\}_{n\geq 1}$

$\gamma, 2\gamma, 3\gamma, 4\gamma, \dots \; n\gamma, \dots \leftarrow$

$\langle\gamma\rangle, \langle 2\gamma\rangle, \langle 3\gamma\rangle, \langle 4\gamma\rangle, \dots, \langle n\gamma\rangle, \dots$

$\gamma = 3.9$

Let $\underline{\gamma \in \mathbb{Q}}$; $\gamma = \frac{p}{q}$; $p \in \mathbb{Z}$, $\boxed{q > 0 \,;\, q \in \mathbb{Z}}$

$n = 2q$

$\underbrace{\langle \frac{p}{q} \rangle}_{\in[0,1)}$; $\underbrace{\langle \frac{2p}{q} \rangle}_{\in[0,1)}$; $\underbrace{\langle \frac{3p}{q} \rangle}_{\in[0,1)}$; $\dots$ $\langle q \cdot \frac{p}{q} \rangle$; $\langle \frac{(q+1)p}{q} \rangle$; $\langle \frac{(q+2)p}{q} \rangle$

$\langle p \rangle$ $\quad \langle p + \frac{p}{q} \rangle$ $\quad \vdots$

$\underline{0} \quad \langle \frac{p}{q} \rangle \quad \langle \frac{2p}{q} \rangle$

Let $\gamma \in \mathbb{I}$: irrational

$\{\langle n\gamma \rangle\}_{n\geq 1}$

$\gamma = \pi = 3.14159\dots$ $\qquad \langle\gamma\rangle = 0.14159\dots$

$2\gamma \approx 6.283\dots \Rightarrow \langle 2\gamma\rangle = 0.283\dots$

$3\gamma \approx 9.4247\dots \qquad \langle 3\gamma\rangle = 0.4247\dots$

$4\gamma \approx 12.5643 \qquad \langle 4\gamma\rangle = 0.5643\dots$

BY CONTRADICTION, ASSUME THAT

$\underbrace{\langle n_1\gamma\rangle = \langle n_2\gamma\rangle}$; $n_1 \neq n_2$; $n_1, n_2 \in \mathbb{N}$

$n_1\gamma \sim n_2\gamma \iff n_1\gamma \equiv n_2\gamma \mod 1$

$\iff n_1\gamma - n_2\gamma \in \mathbb{Z}$

$\iff (n_1 - n_2)\gamma = a$; $a \in \mathbb{Z}$

$n_1 - n_2 \in \mathbb{Z}$

$n_1 - n_2 \neq 0$

$\gamma = \frac{a \in \mathbb{Z}}{n_1 - n_2 \in \mathbb{Z}}$

$\Rightarrow \gamma \in \mathbb{Q}$ contradiction

KRONECKER: $\underline{\{\langle n\gamma\rangle\}_{n\geq 1}}$ is $\underline{\text{dense}}$ in $[0,1)$.

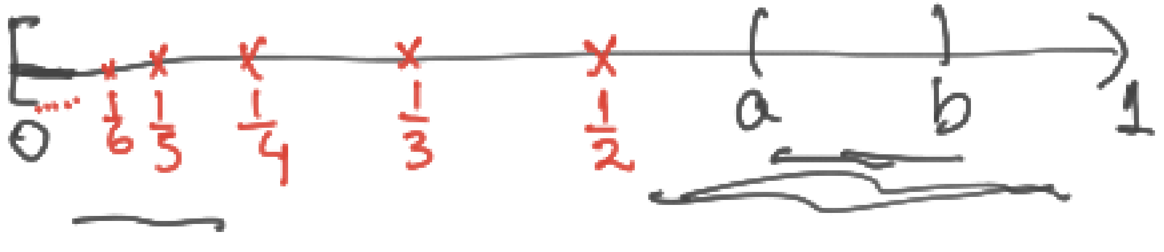
set

$0 \quad \langle n_1\gamma\rangle \quad n_3\gamma \quad \langle n_2\gamma\rangle \quad 1$

DEF: The sequence $\{\xi_n\}_{n\geq 1} \subset [0,1)$ is EQUIDISTRIBUTED if for any interval $(a,b) \subset [0,1)$

$$\lim_{N\to\infty} \frac{\#\{1\leq n < N : \xi_n \in (a,b)\}}{N} = b-a$$

EXAMPLE: $\xi_n = \frac{1}{n+1}$ $\quad b>a$



$$\{1\leq n \leq N : \xi_n \in (a,b)\} = 0$$

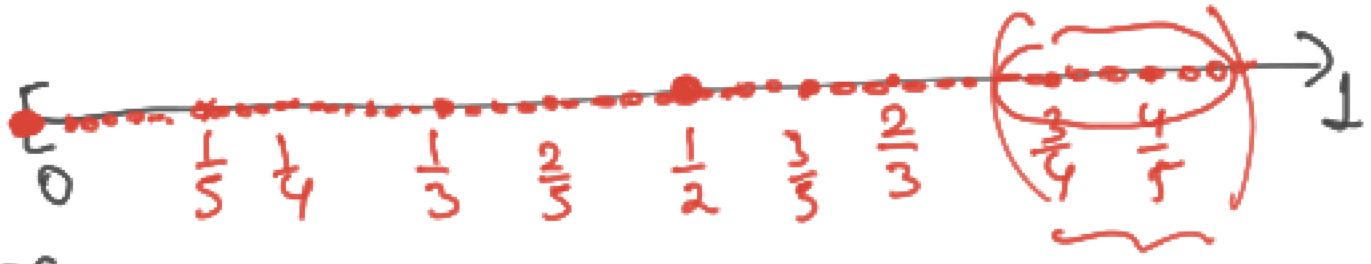
$$\lim_{N\to\infty} \frac{\{1\leq n\leq N : \xi_n \in (a,b)\}}{N} = 0$$

$$b-a = 0 \text{ ?} \longrightarrow \text{FALSE}$$

$\Rightarrow \{\xi_n\}$ is NOT EQUIDISTRIBUTED.

EXAMPLE:

$0;\ \frac{1}{2};\ 0;\ \frac{1}{3};\ \frac{2}{3};\ 0;\ \frac{1}{4};\ \frac{2}{4};\ \frac{3}{4};\ 0, \ldots\ldots$ $\quad \xi_n,\ 1\leq n\leq N$



EXERCISE.

Theorem: Let $\gamma \in I$; $\xi_n = \langle n\gamma \rangle_{n\geq 1} \subset [0,1)$

$\Rightarrow \xi_n$ is EQUIDISTRIBUTED.