Class 17: APPLICATIONS

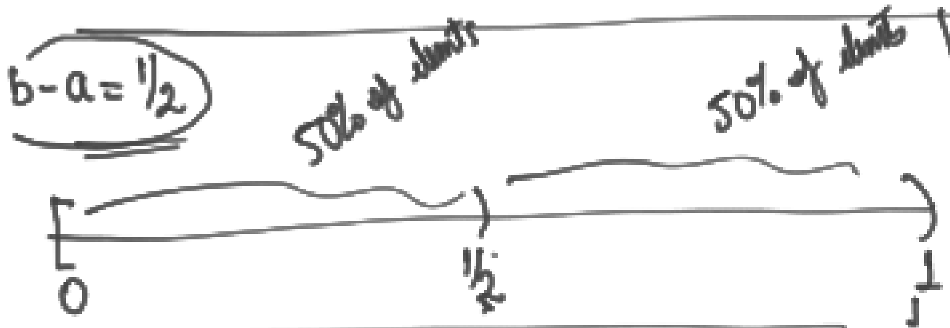
DEFINITION: The sequence $\{\xi_n\}_{n\geq 1} \subset [0,1)$ is EQUIDISTRIBUTED, if

$$\lim_{N\to\infty} \frac{\#\{1\leq n\leq N:\ \xi_n\in(a,b)\}}{N} = b-a,$$

for any interval $(a,b)\subset[0,1)$. Let $(a,b)\subset[0,1)$

N: big $\quad \#\{1\leq n\leq N:\ \xi_n\in(a,b)\}\simeq N(b-a)$

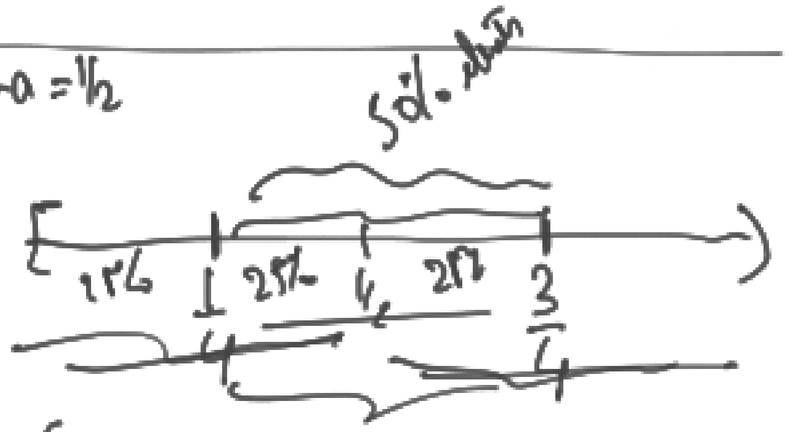
$b-a = 1/2$



$\#\{1\leq n\leq N:\ \xi_n\in(0,\tfrac12)\} = \frac{N}{2} = 50\%N$

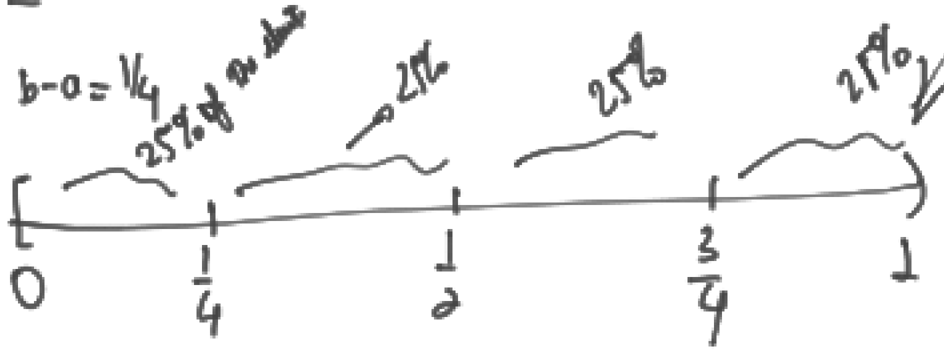
$\#\{1\leq n\leq N:\ \xi_n\in(\tfrac12,1)\} = \frac{N}{2} = 50\%N$

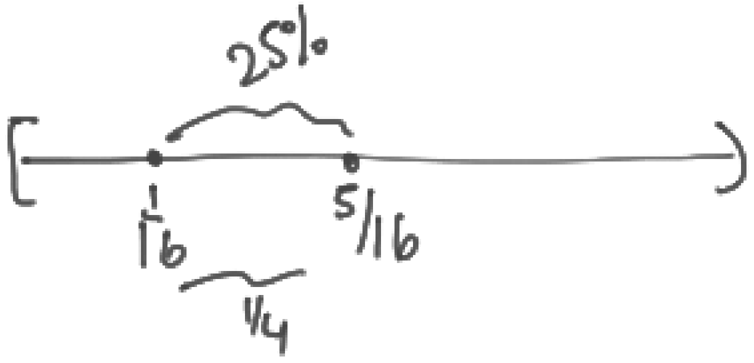
$b-a = 1/2$



$\#\{1\leq n\leq N:\ \xi_n\in(\tfrac14,\tfrac34)\}\simeq N\frac12 = 50\%$

$b-a = 1/4$





$\frac{N}{4} = 25\%N$

Theorem: Let γ be an irrational number. Then $\{\langle n\gamma\rangle\}_{n\geq1}$ is EQUIDISTRIBUTED.

Let $x\in\mathbb{R}$

$$\langle x+1\rangle = x+1-[\![x+1]\!] = x+1-([\![x]\!]+1) = x-[\![x]\!] = \langle x\rangle$$

$\langle x+1\rangle = \langle x\rangle \Rightarrow \langle\cdot\rangle:\mathbb{R}\to[0,1)$ 1-periodic

g is 1-periodic; $\quad f(x) = g\left(\frac{x}{2\pi}\right)$

$$f(x+2\pi) = g\left(\frac{x+2\pi}{2\pi}\right) = g\left(\frac{x}{2\pi}+1\right) = g\left(\frac{x}{2\pi}\right)$$

$$f(x+2\pi) = f(x)$$

f is 2π-periodic $\rightsquigarrow g(x) = f(2\pi x)$

$$g(x+1) = g(x)$$

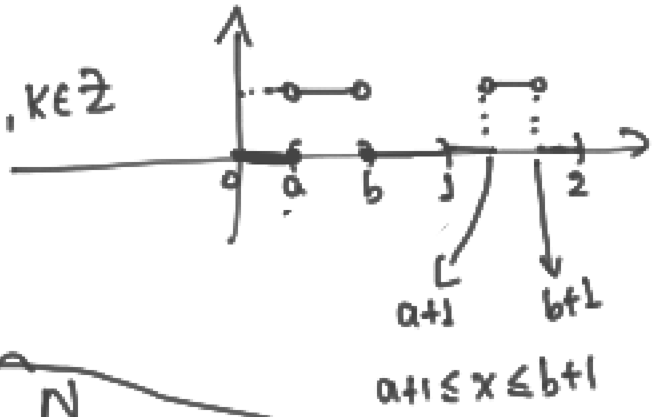
If h is 1-periodic

$$\begin{cases} \boxed{\hat{h}(n) = \int_0^1 h(x)e^{-2\pi i n x}\,dx} \\ P_N(x) = \sum_{|n|\leq N} a_n e^{2\pi i n x} \end{cases}$$

$\#\{1\le n\le N : \langle n\gamma\rangle \in [a,b)\}$

Let $\chi_{[a,b)} : \mathbb{R} \longrightarrow \mathbb{R}$

$$\chi_{[a,b)}(x) = \begin{cases} 1; & x\in[a+k,\,b+k),\ k\in\mathbb{Z} \\ 0; & \text{otherwise} \end{cases}$$



$\chi_{[a,b)}$ is 1-periodic

$$\#\{1\le n\le N : \langle n\gamma\rangle \in [a,b)\} = \sum_{n=1}^{N} \chi_{[a,b)}(n\gamma) \checkmark$$

$$\underline{\langle n\gamma\rangle \in [a,b)} \Leftrightarrow \langle n\gamma\rangle + [\![n\gamma]\!] \in \left(a+[\![n\gamma]\!],\ b+[\![n\gamma]\!]\right)$$

$$n\gamma \in \left(a+[\![n\gamma]\!],\ b+[\![n\gamma]\!]\right)$$

$$\chi_{[a,b)}(n\gamma) = 1$$

$$\frac{\#\{1\le n\le N : \langle n\gamma\rangle \in [a,b)\}}{N} = \frac{\sum_{n=1}^{N} \chi_{[a,b)}(n\gamma)}{N}$$

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \chi_{[a,b)}(n\gamma)}{N} = \int_0^1 \chi_{[a,b)}(y)\,dy$$

WE NEED TO PROVE

Lemma: Let f be an 1-periodic continuous function.

Then:

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} f(n\gamma)}{N} = \int_0^1 f(x)\,dx \quad \ldots (*)$$

PROOF: Let $k\in\mathbb{Z}$: $e_k(x) = e^{2\pi i k x}$, $e_0 = 1$

$k=0 \Rightarrow$

$$\frac{\sum_{n=1}^{N} e_0(n\gamma)}{N} = \frac{\sum_{n=1}^{N} 1}{N} = \frac{N}{N} = 1 = \int_0^1 e_0(x)\,dx$$

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} e_0(n\gamma)}{N} = \int_0^1 e_0(x)\,dx$$

$k\neq 0$

$e_k(x) = e^{2\pi i k x}$, $k \in \mathbb{Z}$, $k \neq 0$

$$S_N = \sum_{n=1}^{N} e_k(n\gamma) = \sum_{n=1}^{N} e^{2\pi i k n \gamma} = \sum_{n=1}^{N} \left(e^{2\pi i k \gamma}\right)^n$$

$$e^{2\pi i k \gamma} S_N = \sum_{n=1}^{N} \left(e^{2\pi i k \gamma}\right)^{n+1} = \sum_{n=2}^{N+1} \left(e^{2\pi i k \gamma}\right)^n = \sum_{n=1}^{N} \left(e^{2\pi i k \gamma}\right)^n - e^{2\pi i k \gamma} + \left(e^{2\pi i k \gamma}\right)^{N+1}$$

$$e^{2\pi i k \gamma} S_N = S_N - e^{2\pi i k \gamma} + \left(e^{2\pi i k \gamma}\right)^{N+1}$$

$$S_N = \frac{\left(e^{2\pi i k \gamma}\right)^{N+1} - e^{2\pi i k \gamma}}{e^{2\pi i k \gamma} - 1} \qquad e^{2\pi i k \gamma} \neq 1$$

$$0 \leq \left| \frac{\sum_{n=1}^{N} e_k(n\gamma)}{N} \right| = \left| \frac{S_N}{N} \right| = \left| \frac{\left(e^{2\pi i k \gamma}\right)^{N+1} - e^{2\pi i k \gamma}}{N\left(e^{2\pi i k \gamma} - 1\right)} \right| \leq \frac{2}{N\left|e^{2\pi i k \gamma} - 1\right|}$$

$N \to \infty$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{e_k(n\gamma)}{N} = 0 = \int_0^1 e_k(x)\,dx$$

(*) HOLDS FOR e_k $\forall k \in \mathbb{Z}$ ✓

$$P_M(x) = \sum_{|k| \leq M} a_k e^{2\pi i k x} \checkmark$$

$$\frac{\sum_{n=1}^{N} P_M(n\gamma)}{N} = \frac{\sum_{n=1}^{N} \left(\sum_{|k| \leq M} a_k e^{2\pi i k (n\gamma)} \right)}{N}$$

$$= \sum_{|k| \leq M} a_k \left(\frac{\sum_{k=1}^{N} e^{2\pi i k n \gamma}}{N} \right)$$

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} P_M(n\gamma)}{N} = \sum_{|k| \leq M} a_k \left(\int_0^1 e_k(x)\,dx \right)$$

$$= \int_0^1 \sum_{|k| \leq M} a_k e_k(x)\,dx$$

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} P_M(n\gamma)}{N} = \int_0^1 P_M(x)\,dx$$

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$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} f(n\gamma)}{N} = \int_0^1 f(x)\,dx \quad \dots \quad (*) \quad ✓$$

(*) HOLDS WHEN f IS A TRIGONOMETRIC POLYNOMIAL.

$$\boxed{\text{Ap. Lemma: For } f \text{ continuous } 1\text{-periodic}, \ \varepsilon>0 \ \exists P \text{ TRIG. POLYNOM. } \ |f(x)-P(x)|<\varepsilon \ \forall x\in\mathbb{R}}$$

Let f continuous, $\varepsilon>0$. $\exists P_\varepsilon$: TRIG. POLYNOMIAL

$$|f(x)-P(x)| < \varepsilon/3 \quad \dots \quad (\alpha)$$

because (*) WORKS FOR P

$\exists N_0 \in \mathbb{N}$: $N \geq N_0$

$$\left| \frac{\sum_{n=1}^{N} P(n\gamma)}{N} - \int_0^1 P(x)\,dx \right| < \frac{\varepsilon}{3}$$

$$\frac{\sum_{n=1}^{N} f(n\gamma)}{N} - \int_0^1 f(x)\,dx = \frac{\sum_{n=1}^{N} f(n\gamma)}{N} - \frac{\sum_{n=1}^{N} P(n\gamma)}{N}$$

$$+ \frac{\sum_{n=1}^{N} P(n\gamma)}{N} - \int_0^1 P(x)\,dx$$

$$+ \int_0^1 P(x)\,dx - \int_0^1 f(x)\,dx$$

$$= I_1 + I_2 + I_3$$

$$\left| \frac{\sum_{n=1}^{N} f(n\gamma)}{N} - \int_0^1 f(x)\,dx \right| \leq |I_1| + |I_2| + |I_3|$$

$$|I_1| \leq \left| \frac{\sum_{n=1}^{N} f(n\gamma) - P(n\gamma)}{N} \right| \leq \frac{\sum_{n=1}^{N} |f(n\gamma) - P(n\gamma)|}{N}$$

$$\overset{(\alpha)}{<} \sum_{n=1}^{N} \frac{\varepsilon}{3N} = \frac{\varepsilon}{3N} N = \frac{\varepsilon}{3}$$

$$|I_1| \leq \varepsilon/3$$

$$I_2 = \frac{\sum_{n=1}^{N} P(n\gamma)}{N} - \int_0^1 P(x)\,dx \Rightarrow |I_2| \leq \varepsilon/3 \; ; \; N \geq N_0$$

$$I_3 = \int_0^1 P(x) - \int_0^1 f(x)\,dx \Rightarrow |I_3| \leq \int_0^1 \underbrace{|P(x) - f(x)|}\,dx$$

$$\leq \int_0^1 \frac{\varepsilon}{3}\,dx = \frac{\varepsilon}{3}$$

$$\underbrace{\left| \frac{\sum_{n=1}^{N} f(n\gamma)}{N} - \int_0^1 f(x)\,dx \right|} \leq |I_1| + |I_2| + |I_3| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

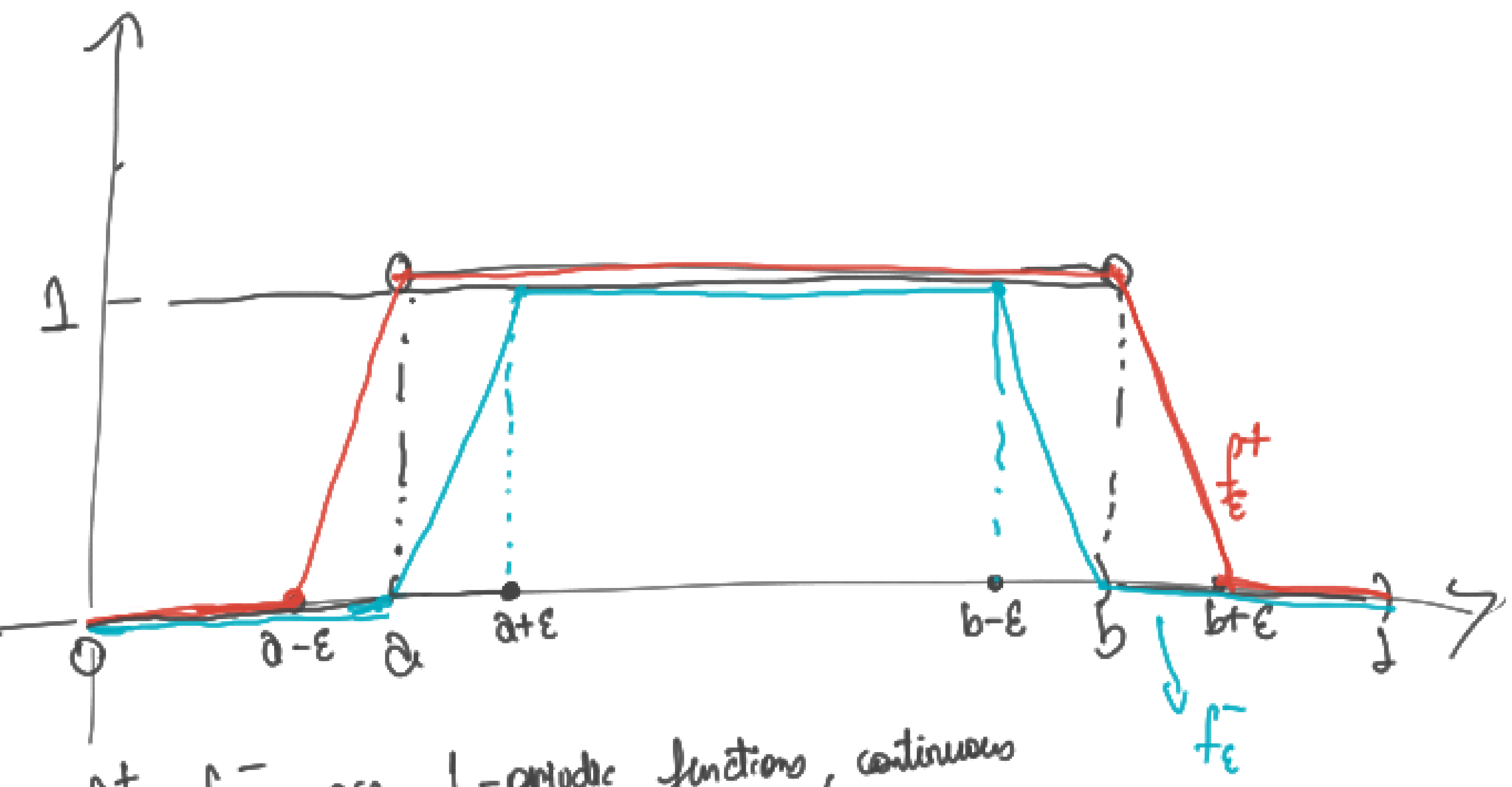
$N \geq N_0$

therefore $\lim_{N\to\infty} \frac{\sum_{n=1}^{N} f(n\gamma)}{N} = \int_0^1 f(x)\,dx$

$\Rightarrow$ (*) WORKS FOR f continuous.

PROOF OF THEOREM:

Let $\varepsilon > 0$ (small)



$f_\varepsilon^+, f_\varepsilon^-$ are 1-periodic functions, continuous

$$f_\varepsilon^-(x) \leq \chi_{(a,b)}(x) \leq f_\varepsilon^+(x) \quad \forall x \in \mathbb{R}$$

$$\int_a^b f_\varepsilon^-(x)\,dx = \frac{b-a+b-\varepsilon-a-\varepsilon}{2} \cdot 1 = b-a-\varepsilon$$

$$\int_a^b f_\varepsilon^+(x)\,dx = \frac{b+\varepsilon-(a-\varepsilon)+b-a}{2} = b-a+\varepsilon$$

$\int_0^1 f_\varepsilon^-(x)dx = b-a-\varepsilon$

$\int_0^1 f_\varepsilon^+(x)dx = b-a+\varepsilon$

$f_\varepsilon^-(x) \le \chi_{(a,b)}(x) \le f_\varepsilon^+(x)$

$$0 \le \underbrace{\frac{\sum_{n=1}^N f_\varepsilon^-(n\gamma)}{N} \le \frac{\sum_{n=1}^N \chi_{(a,b)}(n\gamma)}{N}}_{\downarrow \text{ liminf}} \le \frac{\sum_{n=1}^N f_\varepsilon^+(n\gamma)}{N} \le 1$$

$$\underbrace{\liminf_{N\to\infty} \frac{\sum_{n=1}^N f_\varepsilon^-(n\gamma)}{N}}_{(*) \text{ WORKS FOR } f_\varepsilon^-} \le \liminf_{N\to\infty} \frac{\sum_{n=1}^N \chi_{(a,b)}(n\gamma)}{N}$$

$$\underbrace{\lim_{N\to\infty} \frac{\sum_{n=1}^N f_\varepsilon^-(n\gamma)}{N}}_{\frac{\int_0^1 f_\varepsilon^-(x)dx}{b-a-\varepsilon}} \Rightarrow b-a-\varepsilon \le \liminf_{N\to\infty} \frac{\sum_{n=1}^N \chi_{(a,b)}(n\gamma)}{N}$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\ge n} \{x_k\}\right)$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k\ge n} \{x_k\}\right)$$

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^N \chi_{(a,b)}(n\gamma)}{N} \le \underbrace{\limsup \frac{\sum_{n=1}^N f_\varepsilon^+(n\gamma)}{N}}$$

$$\underbrace{\lim_{N\to\infty} \frac{\sum_{n=1}^N f_\varepsilon^+(n\gamma)}{N}}$$

$$\underbrace{\int_0^1 f_\varepsilon^+(x)dx}$$

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^N \chi_{(a,b)}(n\gamma)}{N} \le b-a+\varepsilon$$

$$b-a \le \underbrace{\liminf_{N\to\infty} \frac{\sum \chi_{(a,b)}}{N}} \le \underbrace{\limsup \frac{\sum \chi_{(a,b)}}{N}} \le b-a$$

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N \chi_{(a,b)}}{N} = b-a = \int_0^1 \chi_{(a,b)}(x)dx$$

$\{\langle n\gamma\rangle\}_{n\geq 1}$ IS EQUIDISTRIBUTED