(23.02.21)

NEWS: – Problem set 3 is out.
↳ Check wiki

– Problem set 4 in ca. 2 weeks?

– Sol'ns to exercises?
↳ Send your sol'ns to prof.
↳ If it's OK, he will upload to wikipage.
↳ ~80% of problems may have their sol'ns uploaded.

– Syllabus has been uploaded.
↳ Check wiki.

$\{\xi_n\}$ is equidistributed if $\forall\ (a,b) \subset [0,1)$.

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N :\ \xi_n \in (a,b)\}}{N} = b-a.$$

$\gamma \in \mathbb{I}$: $\{\langle n\gamma \rangle\}_{n\geq 1}$ is equid?

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{\chi_{(a,b)}(n\gamma)}{N} = \int_0^1 \chi_{(a,b)}(x)\,dx$$

$$(*) \qquad \lim_{N\to\infty} \sum_{n=1}^{\infty} \frac{f(n\gamma)}{N} = \int_0^1 f(x)\,dx$$

¿ (*) works when $f = \chi_{(a,b)}$?

1) (*) $\to$ $e_k = e^{2\pi i k x}$, $k \neq 0$
$ \to$ $e_0 = e^0 \equiv 1$
$\Big\}$ $\Downarrow$ linearity

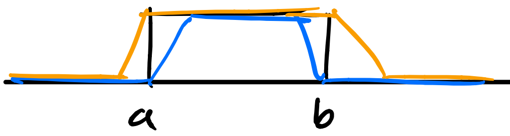
2) (*) WORKS FOR TRIG. POLYNOMIALS

Approximation lemma

3) (*) WORKS for cont. funcs of period 1.

Approximation Lemma

4) (*) Works for $f = \chi_{(a,b)}$. ✓



5) (*) works for any integrable function of period 1.

[Exercise! Tip: use Riemann sums. Upper/Lower sums. See appendix.]

WEYL'S CRITERION:

$\{\xi_n\}_{n\geq 1}$ is equidistributed iff

$$\forall k \in \mathbb{Z},\ k \neq 0 : \quad \lim_{N\to\infty} \frac{\sum_{k=1}^{N} e^{2\pi i k \xi_n}}{N} = 0.$$

Proof:

($\Leftarrow$): Show that 1), 2), 3), 4) above works. It's the same proof.

($\Rightarrow$): Exercise.

Can show that $\{\langle\sqrt{n}\rangle\}_{n\geq 1}$ is equidistributed.

Also: $\{\langle a n^{\sigma}\rangle\}_{n\geq 1}$, $a \neq 0$, $0 < \sigma < 1$, is equidistributed.

BREAK

Fourier Transform

$$\lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx =: \int_{-\infty}^{\infty} f(x)\,dx$$

MODERATE DECREASING FUNCTIONS $\mathcal{M}(\mathbb{R})$

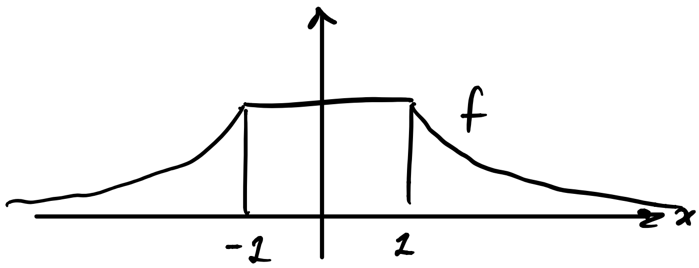
- $f: \mathbb{R} \to \mathbb{C}$ is continuous
- $\underbrace{|f(x)| \leq \frac{M}{|x|^{1+\delta}}}_{\text{DECAY AT INFINTY}}$ for some $\delta > 0$ and $M > 0$, $|x|$ big.

we write $f(x) = O\left(\frac{1}{|x|^{1+\delta}}\right)$.

(E.g.: $|x| \geq 10^{10}$ $|f(x)| \leq \frac{\pi}{|x|^2}$)

E.g.:

(1) $f(x) = \begin{cases} 1, & |x| \leq 1 \\ \frac{1}{|x|^3}, & |x| \geq 1 \end{cases}$



Important: $\int_1^\infty \frac{dx}{x^{1+\delta}}$, $\delta > 0$ exists!

$$\int_1^N \frac{dx}{x^{1+\delta}} = \left.\frac{x^{-\delta}}{-\delta}\right|_1^N = \frac{1}{-\delta}\left\{\frac{1}{N^\delta} - 1\right\} \longrightarrow \frac{1}{\delta} \text{ as } N \to \infty$$

(1) $\displaystyle\int_{-N}^{N} f(x)\,dx = 2\int_0^N f(x)\,dx = 2\Big[\underbrace{\int_0^1 f(x)\,dx}_{\text{just some constant}} + \underbrace{\int_1^N f(x)\,dx}_{\text{important part as } N\to\infty}\Big].$

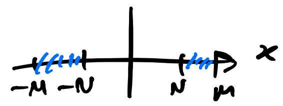
In book: $|f(x)| \le \dfrac{M}{1+x^2}$ or $\le \dfrac{M}{1+|x|^{1+\delta}}$, but we use $\le \dfrac{M}{|x|^{1+\delta}}$.

$f \in M(\mathbb{R})$:

$$\lim_{N\to\infty} \underbrace{\int_{-N}^{N} f(x)\,dx}_{=:\, I_N \,=\, \int_{|x|\le N} f(x)\,dx}$$

$\lim_{N\to\infty} I_N$ exists.

Proof: $\{I_N\}$ is Cauchy sequence.



Let $M>N$:
$$|I_M - I_N| = \left| \int_{|x|\le M} f(x)\,dx - \int_{|x|\le N} f(x)\,dx \right|$$
$$= \left| \int_{N\le|x|\le M} f(x)\,dx \right|$$
$$\le \int_{N\le|x|\le M} |f(x)|\,dx$$
$$\le \int_{N\le|x|\le M} \frac{K}{|x|^{1+\delta}}\,dx \text{ for } \underbrace{\text{some } K}_{\text{and } \delta>0}, \text{ since } f\in M(\mathbb{R})$$

$$\Rightarrow |I_M - I_N| \leq 2K \int_{N \leq |x| \leq M} \frac{dx}{x^{1+\delta}}$$

$$\leq 2K \int_N^\infty \frac{dx}{x^{1+\delta}} = 2K \left[\frac{x^{-\delta}}{-\delta} \Big|_{x=N}^{x=\infty} \right]$$

$$= 2K \left[0 - \left(\frac{N^{-\delta}}{-\delta} \right) \right] = \frac{2K}{\delta} \frac{1}{N^\delta} < \varepsilon \text{ for } N \geq N_0$$

and all $M > N$.

So if $f \in M(\mathbb{R}) \Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx$ is well defined.

$\forall \varepsilon > 0 \quad \exists N_0 \in \mathbb{N} : M > N \geq N_0 :$

$$\Rightarrow |I_M - I_N| < \varepsilon.$$

Let now $M \to \infty \Rightarrow \left| \int_{|x|>N} f(x)\,dx \right| < \varepsilon$, so the "tail" of f must go to zero.

$$! \quad \lim_{N \to \infty} \int_{|x|>N} f(x)\,dx = 0.$$

Proposition:

i) $M(\mathbb{R})$ is a vector space and $f, g \in M(\mathbb{R}) \Rightarrow$

$$\int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x))\,dx = \alpha \int_{-\infty}^{\infty} f(x)\,dx + \beta \int_{-\infty}^{\infty} g(x)\,dx$$

ii) $\displaystyle\int_{-\infty}^{\infty} f(x-h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$

$$\left.\begin{aligned} f(x) &= O\left(\frac{1}{|x|^{1+\delta_1}}\right) \\ g(x) &= O\left(\frac{1}{|x|^{1+\delta_2}}\right) \end{aligned}\right\} \Longrightarrow (f+g)(x) = O\left(\frac{1}{|x|^{1+\min\{\delta_1, \delta_2\}}}\right)$$

Translation function:

$$(\tau_h f)(x) := f(x-h)$$

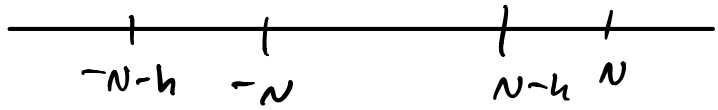
$$\int_{-\infty}^{\infty} (\tau_h f)(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

ii) <u>Proof</u> $\displaystyle\int_{-N}^{N} f(x-h)\,dx = \int_{-N-h}^{N-h} f(\mu)\,d\mu$ (Change of variables.)

$$\lim_{N\to\infty}\left(\int_{-N}^{N} f(x-h)\,dx - \int_{-N}^{N} f(x)\,dx\right) = 0 \ ?$$

$$\int_{-N}^{N} f(x-h)\,dx - \int_{-N}^{N} f(x)\,dx = \int_{-N-h}^{N-h} f(x)\,dx - \int_{-N}^{N} f(x)\,dx$$

$h > 0$



$-N < N-h \Rightarrow h < 2N$