(25.02.21)

CLASS 20: Fourier Transform

$M(\mathbb{R})$: f continuous, and $f(x) = O\left(\frac{1}{|x|^{1+\delta}}\right)$, $\delta > 0$.

Moderately decreasing functions

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx$$

1) $\lim_{N\to\infty} \int_{|x|\geq N} f(x)\,dx = 0.$

$M(\mathbb{R})$ is a vector space: $\int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x))\,dx = \alpha \int_{-\infty}^{\infty} f(x)\,dx + \beta \int_{-\infty}^{\infty} g(x)\,dx$

2) For any $h \in \mathbb{R}$: $\int_{-\infty}^{\infty} \underbrace{f(x-h)}_{(\tau_h f)(x)}\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$

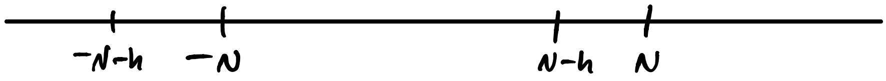
NEW TRICKS NOT IN BOOK.

Proof of 2): Assume $h>0$ and fixed. Let N be big. bigger than $\frac{h}{2}$. big enough that $(*)$ holds

$$\lim_{N\to\infty} \left[\int_{-N}^{N} f(x-h)\,dx - \int_{-N}^{N} f(x)\,dx \right] = 0$$ ← WANT TO PROVE THIS.

Change vars: $x - h =: u$

$$\left| \int_{-N-h}^{N-h} f(u)\,du - \int_{-N}^{N} f(x)\,dx \right| = \left| \int_{-N-h}^{N-h} f(x)\,dx - \int_{-N}^{N} f(x)\,dx \right| = \left| \int_{-N-h}^{-N} f(x)\,dx - \int_{N-h}^{N} f(x)\,dx \right|$$



$$-N < N-h \iff \frac{h}{2} < N$$ Need this. So assume $N > \frac{h}{2}$.

$$\left| \int_{-N-h}^{-N} f(x)\,dx - \int_{N-h}^{N} f(x)\,dx \right| \le \int_{-N-h}^{-N} |f(x)|\,dx + \int_{N-h}^{N} |f(x)|\,dx$$

$\left[f \in M(\mathbb{R}) \iff |f(x)| \le \frac{N}{|x|^{1+\delta}},\ |x| \geq R \text{ some } R \right]$

$$\int_{-N-h}^{-N} |f(x)|\,dx \underset{(*)}{\le} M \int_{-N-h}^{-N} \frac{dx}{|x|^{1+\delta}} = M \int_{N}^{N+h} \frac{dx}{x^{1+\delta}} = M \left[\frac{x^{-\delta}}{-\delta} \Bigg|_{x=N}^{x=N+h} \right]$$

$$= \frac{M}{\delta} \left[\frac{1}{N^\delta} - \frac{1}{(N+h)^\delta} \right]$$

$$= \frac{M}{\delta} \left[\frac{(N+h)^\delta - N^\delta}{N^\delta (N+h)^\delta} \right]$$

$$\le \frac{M}{\delta} \left(\frac{(N+h)^\delta - N^\delta}{N^{2\delta}} \right) \qquad \frac{1}{(N+h)^\delta} \le \frac{1}{N^\delta}$$

$$= \frac{M}{\delta} \frac{1}{N^\delta} \left[\left(\frac{(N+h)}{N} \right)^\delta - \frac{N^\delta}{N^\delta} \right]$$

$$= \frac{M}{\delta} \frac{1}{N^\delta} \left[\left(1 + \frac{h}{N}\right)^\delta - 1 \right] \longrightarrow 0 \text{ as } N \to \infty$$

$$\underbrace{\quad}_{\to 0} \qquad \underbrace{\qquad\qquad}_{\to 1-1=0 \text{ as } N\to\infty}$$

So:

$$0 \leq \left| \int_{-N-h}^{-N} f(x)dx - \int_{N-h}^{N} f(x)dx \right| \leq \underbrace{\int_{-N-h}^{-N} |f(x)|dx}_{\to 0} + \underbrace{\int_{N-h}^{N} |f(x)|dx}_{\to 0} \to 0 \text{ as } N\to\infty.$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x-h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

3) For any $\delta>0$: $$\delta \int_{-\infty}^{\infty} f(\delta x)dx = \int_{-\infty}^{\infty} f(x)dx$$

$\to$ Exercise!

4) $$\lim_{h\to\infty} \int_{-\infty}^{\infty} |\underbrace{f(x-h)-f(x)}_{(\tau_h f)(x) - f(x)}|\,dx = 0$$ "τ is continuous in $L^1(\mathbb{R})$."
τ Translation operator.

Proof: Let $\varepsilon>0$, $|h|\leq h_0$

$\exists h_0\in\mathbb{R}: \Rightarrow \int_{-\infty}^{\infty} |f(x-h)-f(x)|\,dx < \varepsilon.$ WE WANT TO PROVE

"f is uniformly continuous on compact intervals", i.e. $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$.

Let $\varepsilon > 0$, $|h| \leq 1$.

$$\lim_{N\to\infty} \int_{|x|\geq N} f(x)\,dx = 0 \implies \exists N_0, \; N \geq \overset{N_0(\varepsilon)}{N_0} \text{ s.t. } \int_{|x|\geq N} |f(x)|\,dx < \varepsilon$$

$$(\tau_h f)(x) := f(x-h)$$

$$|f(x)| \leq \frac{M}{|x|^{1+\delta}} \Rightarrow |f(x-h)| \leq \frac{M}{|x-h|^{1+\delta}} \leq \frac{M'}{|x|^{1+\delta}}$$

Since $\tau_h(f) \in M(\mathbb{R}) \implies \lim_{N\to\infty} \int_{|x|\geq N} (\tau_h(f))(x)\,dx < \varepsilon$ as well.

$$\int_{|x|\geq N} |f(\underbrace{x-h}_{=:u})|\,dx = \int_{|u+h|\geq N} |f(u)|\,du = \int_{|x+h|\geq N} |f(x)|\,dx \leq \int_{|x|\geq N-1} |f(x)|\,dx$$

$$|u|+|h| \geq |u+h| \geq N$$

$$|x|+|h| \geq |x+h| \geq N$$

$$|x| \geq N - |h| \geq N-1$$

$$\int_a^b |f(x)|\,dx \leq \int_{a-\varepsilon}^{b+\varepsilon} |f(x)|\,dx$$

? N_0+1 : $\displaystyle\int_{|x|\geq N_0+1} |f(x)|\,dx < \varepsilon$, $\displaystyle\int_{|x|\geq N_0+1} |f(x-h)|\,dx < \varepsilon$.

$$\int_{-\infty}^{\infty} |f(x-h)-f(x)|\,dx = \int_{|x|\geq N_0+1} |f(x-h)-f(x)|\,dx + \int_{|x|\leq N_0+1} |f(x-h)-f(x)|\,dx = I_1 + I_2 \leq 4\varepsilon.$$

$$0 \leq I_1 \leq \int_{|x|\geq N_0+1} |f(x-h)|\,dx + \int_{|x|\geq N_0+1} |f(x)|\,dx < 2\varepsilon.$$

For I_2: f is uniformly continuous in $[-(N_0+2), N_0+2]$.

$\Rightarrow \exists\, \underset{h_1(\varepsilon)}{h_1} > 0$: $z, y \in [-(N_0+2), N_0+2]$

$$|z-y| < h_1 \Rightarrow |f(z)-f(y)| < \varepsilon$$

Now: $z := x-h$ and $y := x$, so that:

$x-h, x \in [-(N_0+2), N_0+2]$:

$$|h| < h_1 \Rightarrow |f(x-h)-f(x)| < \frac{\varepsilon}{N_0+1}.$$

$$0 \leq I_2 \leq \int_{|x|\leq N_0+1} \frac{\varepsilon}{N_0+1}\,dx \quad \text{when } |h| \leq h_0 \quad \text{Take } h_0 \leq 1\ ?$$

$$= \frac{\varepsilon}{N_0+1} \cdot 2(N_0+1) = 2\varepsilon.$$

So we are done.

BREAK

Seminar page: Researchseminars.org

FOURIER TRANSFORM

Let $f \in M(\mathbb{R})$:

$\xi \in \mathbb{R}$: $\quad \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx$ is the Fourier transform of f.

$\hat{f} : \mathbb{R} \to \mathbb{C}$

$E(x) = f(x) e^{-2\pi i x \xi}$

$|E(x)| = |f(x)|$

$$|\hat{f}(\xi)| = \left| \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx \right| \leq \int_{-\infty}^{\infty} |f(x)| dx$$

i) $\hat{f}$ is a bounded function.

ii) $\hat{f}$ is continuous

Fix $\xi \in \mathbb{R}$: $\lim_{h \to 0} \hat{f}(\xi + h) - \hat{f}(\xi) = 0.$

$$\hat{f}(\xi + h) - \hat{f}(\xi) = \int_{-\infty}^{\infty} \left(f(x) e^{-2\pi i x(\xi + h)} - f(x) e^{-2\pi i x \xi} \right) dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \cdot \left(e^{-2\pi i x h} - 1 \right) dx$$

$h \neq 0$:

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)| \cdot |e^{-2\pi i x h} - 1| \, dx$$

$$= \int_{|x| \geq 1/\sqrt{|h|}} |f(x)| |e^{-2\pi i x h} - 1| \, dx + \int_{|x| \leq 1/\sqrt{|h|}} |f(x)| |e^{-2\pi i x h} - 1| \, dx$$

$$= \quad I_1 \quad + \quad I_2$$

so: $xh \leq \sqrt{h}$
?

Let $\varepsilon > 0$:

$0 \leq I_1 \leq 2 \int_{|x| \geq N} |f(x)| \, dx < \varepsilon$, $\forall N \geq N_0$. for some $N_0 = N_0(\varepsilon)$.

If h satifies $\frac{1}{\sqrt{h}} \geq N_0 \iff \frac{1}{N_0} \geq \sqrt{h}$

$$I_1 \leq 2 \int_{|x| \geq 1/\sqrt{|h|}} |f(x)| \, dx \leq 2 \int_{|x| \geq N_0} |f(x)| \, dx < \varepsilon$$

$$\{x : |x| \geq \tfrac{1}{\sqrt{h}}\} \subset \{x : |x| \geq N_0\}.$$

1) IT MEANS: $|h| \leq \left(\frac{1}{N_0}\right)^2 \Rightarrow I_1 < \varepsilon$ ✓

2) $|x| \leq \frac{1}{\sqrt{|h|}}$: $|e^{-2\pi i x h} - 1|^2 = |\cos(2\pi x h) - i \sin(2\pi x h) - 1|^2$

(*)

$$= |\cos(2\pi x h) - 1 - i \sin(2\pi x h)|$$

$$= \ldots$$

$$= 2 - 2\cos(2\pi x h)$$
$$= 2(1-\cos(2\pi x h))$$
$$= 2\cdot 2\sin^2(\pi x h)$$

$\cos 2\vartheta = \cos^2\vartheta - \sin^2\vartheta$

$\cos 2\vartheta = 1 - 2\sin^2\vartheta$

$2\sin^2\vartheta = 1-\cos 2\vartheta$

$$\Rightarrow \left|e^{-2\pi i x h}-1\right|^2 = 4\sin^2(\pi x h).$$

$$\Leftrightarrow \left|e^{-2\pi i x h}-1\right| \leq 2\left|\sin(\pi x h)\right| \leq 2\pi x h$$
$$= 2\pi |h|\cdot|x|.$$
in (*)
$$\leq 2\pi |h|\cdot \frac{1}{\sqrt{|h|}}$$
$$= 2\pi\sqrt{|h|}$$

$y \in \mathbb{R}$: $\sin y = \int_0^y (\sin t)'\,dt = \int_0^y \cos t\,dt$

$$|\sin y| \leq \int_0^{|y|} |\cos t|\,dt \leq \int_0^{|y|} dt = |y|$$

$$I_2 = \int_{|x|\leq \frac{1}{\sqrt{|h|}}} |f(x)|\cdot\left|e^{-2\pi i x h}-1\right| dx \leq 2\pi\sqrt{h}\int_{|x|\leq 1/\sqrt{|h|}} |f(x)|\,dx$$
$$\leq 2\pi\sqrt{|h|}\underbrace{\int_{-\infty}^{\infty} |f(x)|\,dx}_{=:M}$$
$$\leq 2\pi\sqrt{|h|}\cdot M.$$

$$2\pi\sqrt{|h|}\,M < \varepsilon \Leftrightarrow |h| < \left(\frac{\varepsilon}{2\pi M}\right)^2$$

For $|h| \leq \underbrace{\min\left\{\left(\frac{1}{N_0}\right)^2, \left(\frac{\varepsilon}{2\pi M}\right)^2\right\}}_{=:h_0}$.

$$0 \leq I_1 < \varepsilon$$

$$0 \le I_2 < \varepsilon$$

$$\Longrightarrow \quad |\hat{f}(\xi + h) - \hat{f}(\xi)| < 2\varepsilon .$$

so $\hat{f}$ is continuous on $\mathbb{R}$.