(02.03.21)

CLASS 22

NEWS:

* Thursdays : 16 – 17 [4-5pm] Zoom Exercise Class
  ↳ Send message on email. ◡̈
* Please Complete problem set 3 this week!

$f \in M(\mathbb{R})$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx, \quad \xi \in \mathbb{R}. \quad \text{So } \hat{f}: \mathbb{R} \to \mathbb{C}.$$

*) $\hat{f}(\xi)$ is bounded.

$$|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)| \, dx \quad \text{Proved last time}$$

*) $\hat{f}$ is continuous on $\mathbb{R}$.

↘ Uniformly cont.: $|x-y| < \delta \Rightarrow |\hat{f}(x) - \hat{f}(y)| < \varepsilon$.

$|h| < \delta \Rightarrow |\hat{f}(\xi+h) - \hat{f}(\xi)| < \varepsilon$

$$\left| \hat{f}(\xi+h) - \hat{f}(\xi) \right| \le \int_{-\infty}^{\infty} \left| f(x) \right| \underbrace{\left| e^{-2\pi i x \xi} \right|}_{=1} \left| \left(e^{-2\pi i h x} - 1 \right) \right| dx$$

*) Riemann–Lebesgue LEMMA:

For series: $\hat{f}(k) \to 0$ as $k \to \pm\infty$.

Now: $\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0$. ← EXERCISE! Hint: Change of variables.

~~EXAMPLE:~~ $f(\xi) = \chi_{[-a,a]}(\xi)$ NOPE: Not continuous!

1) Let $h \in \mathbb{R}$, $f \in M(\mathbb{R})$

$$(\tau_{-h} f)(x) = f(x+h)$$

$$\widehat{\tau_{-h}(f)}(\xi) = \int_{-\infty}^{\infty} (\tau_h f)(x) e^{-2\pi i x \xi} dx$$

$$= \int_{-\infty}^{\infty} f(x+h) e^{-2\pi i (x+h)\xi} e^{2\pi i h \xi} dx$$

$$= e^{2\pi i h\xi} \int_{-\infty}^{\infty} f(x+h) e^{-2\pi i (x+h)\xi} dx$$

Have proved this "change of variables"

$$= e^{2\pi i h \xi} \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

Fix $\xi \in \mathbb{R}$:

$\underbrace{f(x) e^{-2\pi i x \xi}}_{\in M(\mathbb{R})}$ is also moderately decreasing

And $\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty} g(x+h)\,dx$

$$= e^{2\pi i h \xi} \hat{f}(\xi).$$

So: $\widehat{\tau_{-h} f}(\xi) = e^{2\pi h \xi} \hat{f}(\xi)$, $(\tau_{-h} f)(x) = f(x+h)$

Very strong result!

2) Let $h \in \mathbb{R}$, $f \in M(\mathbb{R})$

$$g_h(x) = e^{-2\pi i x h} f(x).$$

$$\hat{g}_h = ?$$

$$\hat{g}_h(\xi) = \int_{-\infty}^{\infty} g_h(x) e^{-2\pi i x \xi} dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x h - 2\pi i x \xi} dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x (h+\xi)} dx$$

So: $\hat{g}_h(\xi) = \hat{f}(\xi + h)$.

So: $\left(\widehat{e^{-2\pi i x h} f}\right)(\xi) = \hat{f}(\xi + h) = (\tau_{-h} \hat{f})(\xi)$.

Also very important property!

3) Let $\delta > 0$:

$$(D_\delta f)(x) = f(\delta x)$$

$$\widehat{D_\delta f}(\xi) = \int_{-\infty}^{\infty} (D_\delta f)(x) e^{-2\pi i \xi x} dx$$

$$= \int_{-\infty}^{\infty} f(\delta x) e^{-2\pi i \xi x} dx$$

$$= \frac{\delta}{\delta} \int_{-\infty}^{\infty} f(\delta x)\, e^{-2\pi i \frac{\xi}{\delta}(\delta x)}\, dx$$

$$\delta \int_{-\infty}^{\infty} f(\delta x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

$$= \frac{1}{\delta} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \frac{\xi}{\delta} x}\, dx$$

$$= \frac{1}{\delta} \hat{f}\left(\frac{\xi}{\delta}\right)$$

So: $\left(\widehat{f(\delta x)}\right)(\xi) = \frac{1}{\delta} \hat{f}\left(\frac{\xi}{\delta}\right)$.

4) • $f \in M(\mathbb{R}) \cap C^1(\mathbb{R})$

• $f' \in M(\mathbb{R})$.

$\widehat{f'}(\xi)$:

int. by parts

$$I_N = \int_{-N}^{N} f'(x)\, e^{-2\pi i x \xi}\, dx = f(x) e^{-2\pi i x \xi} \Big|_{x=-N}^{x=+N} - \int_{-N}^{N} f(x) \left(e^{-2\pi i x \xi}\right)' dx$$

$$I_N(\xi) = f(N) e^{-2\pi i N \xi} - f(-N) e^{2\pi i N \xi} - \int_{-N}^{N} f(x)(-2\pi i \xi) e^{-2\pi i x \xi}\, dx$$

$$= f(N) e^{-2\pi i N \xi} - f(-N) e^{2\pi i N \xi} + 2\pi i \xi \int_{-N}^{N} f(x)\, e^{-2\pi i x \xi}\, dx$$

$|f(x)| \le \frac{C}{|x|^{1+\delta}}$ so: $|f(N)e^{-2\pi i \xi N}| = |f(N)| \cdot 1 \le \frac{C}{N^{1+\delta}}$

As $N \to \infty \Rightarrow f(N)e^{-2\pi i \xi N} \xrightarrow[N\to\infty]{} 0$.

As $N \to \infty$: I_N: $\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$ $\dashv$

Very important!

Solve DIFF EQNS: $\widehat{f'_{(x)}} = \widehat{e^{-\pi x^2}}$

$2\pi i \xi \hat{f} = \widehat{e^{-\pi x^2}}$

$\hat{f'}_{(x)} = \widehat{2\pi i x f(x)}$

$2\pi i \xi \hat{f} = -(\hat{f}(\xi))'$

BREAK

5) $f \in M(\mathbb{R})$

$xf \in M(\mathbb{R})$

$\Rightarrow \hat{f}$ is differentiable and $\widehat{-2\pi i x f}(\xi) = (\hat{f})'(\xi)$

Proof: Let $\xi \in \mathbb{R}, h \in \mathbb{R}, \varepsilon > 0$:

$$\left| \frac{\hat{f}(\xi+h) - \hat{f}(\xi)}{h} - \widehat{(-2\pi i x f)}(\xi) \right| =$$

$$= \left| \int_{-\infty}^{\infty} f(x) \frac{e^{-2\pi i x(\xi+h)} - e^{-2\pi i x\xi}}{h} dx - \int_{-\infty}^{\infty} (-2\pi i x f(x)) e^{-2\pi i x \xi} dx \right|$$

$$= \left| \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \left\{ \frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right\} dx \right|$$

$$\leq \int_{-\infty}^{\infty} |f(x)| \cdot \left| \frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x \right| dx = \int_{|x| \geq N} + \int_{|x| \leq N} = I_1 + I_2.$$

$\leq 4\pi\varepsilon$

We have: $\exists N_0 : N \geq N_0 : \int_{|x| \geq N} |f(x)| dx < \varepsilon, \quad \int_{|x| \geq N} |x f(x)| dx < \varepsilon.$

$$I_1 \leq \int_{|x| \geq N} |f(x)| \left\{ \left| \frac{e^{-2\pi i x h} - 1}{h} \right| + |2\pi x| \right\} dx = \int_{|x| \geq N} |f(x)| \left| \frac{e^{-2\pi i x h} - 1}{h} \right| dx + 2\pi\varepsilon$$

$$\begin{aligned}
\left| e^{-2\pi i x h} - 1 \right|^2 &= \left| \cos(2\pi x h) - 1 - i \sin(2\pi x h) \right|^2 \\
&= (\cos(2\pi x h) - 1)^2 + \sin^2(2\pi x h) \\
&= 2 - 2\cos(2\pi x h) \\
&= 2(2\sin^2(\pi x h)) \\
&= 4\sin^2(\pi x h)
\end{aligned}$$

$$\left| e^{-2\pi i x h} - 1 \right| = 2|\sin(\pi x h)| \leq 2|\pi x h|$$

So $$I_1 \le \int_{|x|\ge N} |f(x)| \left| \frac{e^{-2\pi i x h} - 1}{h} \right| dx + 2\pi\varepsilon$$

$$\le \int_{|x|\geq N} |f(x)| \cdot 2\pi x \, dx + 2\pi\varepsilon \le 4\pi\varepsilon .$$

$$I_2 \le \int_{|x|\le N_0} |f(x)| \left| \frac{e^{-2\pi x h} - 1 + 2\pi i x h}{h} \right| dx$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ so: $e^{-2\pi i x h} = 1 - 2\pi i x h + \frac{(-2\pi i x h)^2}{2!} + \ldots$

$\omega = -2\pi i x h$:

$e^{\omega} - 1 - \omega \approx \omega^2$:

$$\lim_{\omega \to 0} \frac{e^{\omega}-1-\omega}{\omega^2} \overset{\text{L'Hopital}}{=} \lim_{\omega\to 0} \frac{e^{\omega}-1}{2\omega} = \lim_{\omega \to 0} \frac{e^{\omega}}{2} = \frac{1}{2} .$$

$$\Rightarrow \left| \frac{e^{\omega}-1-\omega}{\omega^2} \right| \le 1 \quad \text{when } |\omega| \le \delta . \quad \left(\text{Use: } \Leftarrow \left| \frac{e^{\omega}-1-\omega}{\omega^2} - \frac{1}{2} \right| < \varepsilon \right)$$

ε = something.

So: $0 < |\omega| \le \delta$:

$$\left| \frac{e^{-2\pi i x h} - 1 + 2\pi i x h}{\underbrace{(-2\pi i x h)^2}_{4\pi^2 x^2 h^2}} \right| \le 1 . \qquad 0 \le |-2\pi i x h| \le \delta$$

So: $$\left| e^{-2\pi i x h} - 1 + 2\pi i x h \right| \le 4\pi^2 x^2 h^2 , \quad 0 \le |xh| \le \frac{\delta}{2\pi} .$$

Remember: $N_0 = N_0(\varepsilon)$.

I_2: If you take $|h| \le \frac{\delta}{2\pi N_0} \Rightarrow |xh| \le \frac{\delta}{2\pi N_0} \underbrace{|x|}_{|x| \le N_0} \le \frac{\delta}{2\pi}$

$$I_2 = \int_{|x| \le N_0} |f(x)| \left| \frac{e^{-2\pi i x h} - 1 + 2\pi i x h}{h} \right| dx \le \int_{|x| \le N_0} |f(x)| \, 4\pi^2 x^2 |h| \, dx$$

$$\le |h| 4\pi^2 N_0^2 \int_{|x| \le N_0} |f(x)| dx$$

$$\le |h| \cdot 4\pi^2 N_0^2 \int_{-\infty}^{\infty} |f(x)| dx$$

$$\le |h| 4\pi N_0^2 B$$

$0 \le I_2 \le |h| 4\pi N_0^2 B \le \varepsilon$, $|h| \le \frac{\varepsilon}{4\pi^2 N_0^2 B}$

and $|h| \le \delta / 2\pi N_0$

$\Rightarrow |h| \le h_0(\varepsilon)$.

$0 \le I_1 + I_2 \le 4\pi\varepsilon + \varepsilon = (4\pi + 1)\varepsilon$.