(04.03.21)

Class 23: Schwartz Space

$f \in M(\mathbb{R})$

i) $f \in C^1(\mathbb{R})$, $f' \in M(\mathbb{R})$

$$\Rightarrow \widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$$

ii) $xf \in M(\mathbb{R})$

$$\Rightarrow \left(\widehat{-2\pi i x f}\right)(\xi) = (\hat{f})'(\xi)$$

Never forget this property! Use it a lot.

$S(\mathbb{R})$: $f \in C^\infty(\mathbb{R})$ and for any $k \geq 0$, $\ell \geq 0$: integers

$$\sup_{x \in \mathbb{R}} \underbrace{|x|^k |f^{(\ell)}(x)|}_{\text{so this is bounded.}} < \infty.$$

$f \in S(\mathbb{R}) \Rightarrow f \in M(\mathbb{R})$:

$$|x|^2 |f(x)| \leq M \Rightarrow |f(x)| \leq \frac{M}{|x|^2}, \quad x \neq 0.$$

$\rightarrow$ $f'(x) \in M(\mathbb{R})$:

$$|x|^2 |f'(x)| \leq N \Rightarrow |f'(x)| \leq \frac{M}{|x|^2}, \quad x \neq 0.$$

$$\Rightarrow \quad |x|^3 |f(x)| \leq K \Rightarrow |x f(x)| \leq \frac{K}{|x|^2}, \quad x \neq 0.$$

[Polynomials not in $S(\mathbb{R})$: $x^2 \notin S(\mathbb{R})$!]

But the Gaussian is.

Let $a>0$: $G(x) = e^{-ax^2}$.

$$\sup_{x \in \mathbb{R}} |x|^k e^{-ax^2} < \infty \,? \qquad k \geq 0.$$

$x \in [-M, M]$ ok b/c it's cont on this interval.

$|x| \geq M$ important case!

L'Hopital k times

$$\lim_{x \to \infty} x^k e^{-ax^2} = \lim_{x \to \infty} \frac{x^k}{e^{ax^2}} = \lim \frac{k x^{k-1}}{e^{ax^2} 2ax}$$

$$= \dots$$

$$= \lim_{x \to \infty} \frac{c}{e^{ax^2} P(x)} = 0.$$

So supremum exists.

L'Hopital

$$\lim_{x \to \infty} x^k \left(e^{-ax^2}\right)^{(\ell)} = \lim_{x \to \infty} \frac{Q(x)}{e^{ax^2}} = \dots = 0$$

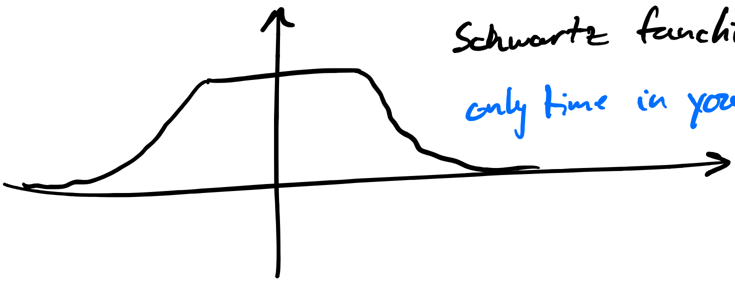
So: $e^{-ax^2} \in S(\mathbb{R})$.

Remember:

$$e^{-x^2} \leq \frac{M}{x^k} \quad \text{for any fixed } k \text{, for large } |x|.$$

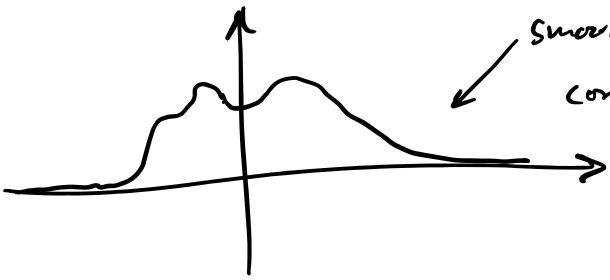
And $e^{x^2} \geqslant M x^k$ ______ " ______ .

Exercises:



Schwartz functions.

only time in your life when you will construct this.



smooth bend.

compact support.

Can construct this.

Prop: $f \in S(\mathbb{R}) \implies \hat{f} \in S(\mathbb{R})$.

Let $k \geqslant 0$, $\ell \geqslant 0$, $k, \ell \in \mathbb{Z}$.

$\hat{f} \in C^\infty(\mathbb{R})$:

i) $\widehat{f'} = 2\pi i \xi \hat{f}$

ii) $\widehat{-2\pi i x f} = (\hat{f})'$

$$x^{k+2} f \Rightarrow \quad x^k f \le \frac{1}{x^2} \Rightarrow \hat{f} \in C^\infty(\mathbb{R})$$

$|\xi|^k (\hat{f})^{(\ell)}$ is bounded?

$$K(x) := \left((-2\pi i x)^\ell f \right)^{(k)} (x)$$

$$\hat{K}(\xi) = \widehat{\left((-2\pi i x)^\ell f \right)^{(k)}} (\xi)$$

$$\overset{\text{by i)}}{=} (2\pi i \xi)^k \widehat{\left[(-2\pi i x)^\ell f \right]} (\xi)$$

$$= (2\pi i \xi)^k (\hat{f})^{(\ell)} (\xi)$$

$$= (2\pi i)^k \xi^k (\hat{f})^{(\ell)} (\xi) \quad \text{is bounded.}$$

because $\hat{f} \in C^\infty$.

$$G(x) = e^{-\pi x^2} \quad ; \quad \int_{-\infty}^{\infty} G(x)\, dx = 1 :$$

$$\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx \right)^2 = \left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-\pi y^2} dy \right)$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\pi (x^2+y^2)} dx \right) dy$$

Polar coord: $x = r\cos\vartheta$, $y = r\sin\vartheta$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-\pi r^2}\, r \, dr \, d\vartheta$$

$$= \int_0^{2\pi} \underbrace{\int_0^{\infty} \left(\frac{e^{-\pi r^2}}{-2\pi} \right)' dr}\, d\vartheta$$

$$= \int_0^{2\pi} \left(\frac{e^{-\pi r^2}}{-2\pi} \right) \Bigg|_{r=0}^{\infty} d\vartheta$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \, d\vartheta = \frac{2\pi}{2\pi} = \underline{\underline{1}}.$$

[Can do this in $\mathbb{R}^n$ also, not just $\mathbb{R}$. Harmonic Analysis is next Fourier course.]

<u>Prop</u>: $\hat{G}(\xi) = G(\xi)$.

So: $G(x)$ is A FIXED POINT OF THE FOURIER TRANSFORM.

i) $\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$

ii) $\widehat{-2\pi i x f}(\xi) = (\hat{f})'(\xi)$

$$F(\xi) := \int_{-\infty}^{\infty} G(x)\, e^{-2\pi i x \xi} \, dx = \hat{G}(\xi)$$

$$F'(\xi) = \int_{-\infty}^{\infty} -2\pi i x\, G(x)\, e^{-2\pi i x \xi}\, dx$$ by ii) ?

$$G(x) = e^{-\pi x^2} \Rightarrow G'(x) = e^{-\pi x^2}(-2\pi x)$$

$$iG'(x) = G(x)(-2\pi i x)$$

$$F'(\xi) = i\int_{-\infty}^{\infty} G'(x)\, e^{-2\pi i x \xi}\, dx = i\,\widehat{G'}$$

$$\overset{\text{by } i)}{=} i\, 2\pi i \xi\, \hat{G}(\xi)$$

$$= -2\pi\xi F(\xi)$$

Have now a diff. eq'n:

$$F'(\xi) = -2\pi\xi F(\xi)$$

DEFINE:

$$M(\xi) := e^{\pi\xi^2} F(\xi) \Rightarrow M'(\xi) = e^{\pi\xi^2}\Big(\underbrace{2\pi\xi F(\xi) + F'(\xi)}_{=0}\Big)$$

$$M'(\xi) = 0$$

$$\Rightarrow M(\xi) = \text{constant.}$$

$$M(0) = F(0) = 1$$ [b/c $\int_{-\infty}^{\infty} G(x)dx = 1.$]

$$e^{\pi\xi^2} F(\xi) = 1 \Rightarrow \hat{G}(\xi) = e^{-\pi\xi^2} = G(\xi). \blacksquare$$

BREAK

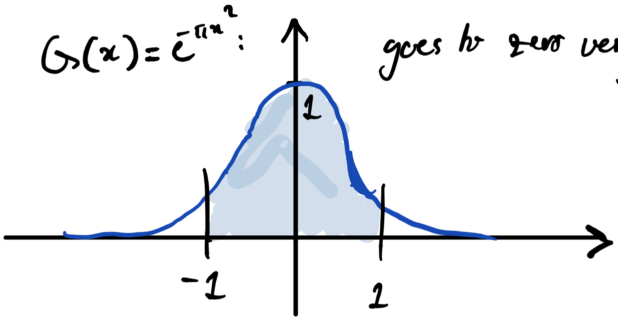
Let $\delta > 0$: $K_\delta(x) = \delta^{-1/2} e^{-\frac{\pi x^2}{\delta}} = \delta^{-\frac{1}{2}} e^{-\pi(x\delta^{-1/2})^2}$

$= \delta^{-1/2} G(\delta^{-1/2} x)$

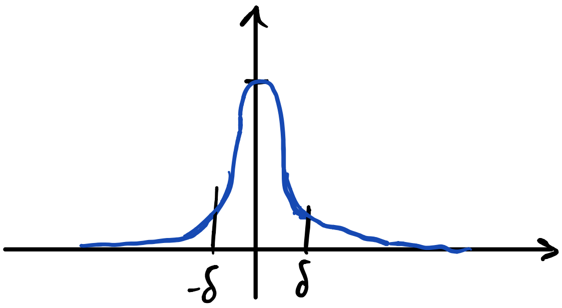
$$\Rightarrow \hat{K}_\delta(\xi) = e^{-\pi\left(\frac{x}{\delta^{-1/2}}\right)^2} = e^{-\pi\delta x^2}$$

$\eta \cdot \widehat{\hat{f}(\eta x)} = \hat{f}\left(\frac{x}{\eta}\right)$, $\eta > 0$

$G(x) = e^{-\pi x^2}$: goes to zero very fast.

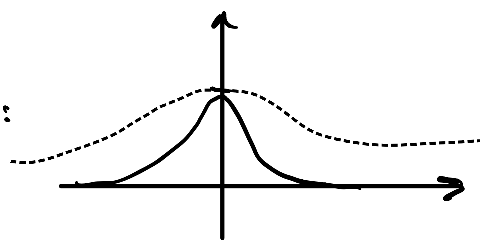


$\delta < 1$: K_δ:



$$\int_{-\infty}^{\infty} K_\delta(x)\,dx = \hat{K}_\delta(0) = 1.$$

$\delta \to 0$:



UNCERTAINTY PRINCIPLE

f and $\hat{f}$ cannot both be concentrated at the same time.

1) $$\int_{-\infty}^{\infty} K_\delta(x)\,dx = \widehat{K_\delta}(0) = 1$$

2) $$\int_{-\infty}^{\infty} |K_\delta(x)|\,dx \leq M \qquad \text{for any } \delta \gtrless 0.$$

Let $\eta > 0$: $\delta \to 0$:

3) $$\int_{|x|\geq\eta} K_\delta(x)\,dx = \int_{|x|\geq\eta} \delta^{-1/2} G(\delta^{-1/2} x)\,dx$$

$$\omega := \delta^{-1/2} x \quad = \int_{|\omega|\geq \eta/\delta^{1/2}} G(\omega)\,d\omega \longrightarrow 0 \quad \text{as } \delta \to 0$$

$$\lim_{M\to\infty} \int_{|x|\geq M} e^{-\pi x^2}\,dx = 0 \quad \text{b/c } G \in M(\mathbb{R})$$

$$\left[\int_{|x|\geq M} G(x)\,dx \leq \int_{|x|\geq [M]-1} G(x)\,dx \right]$$

$[M]$: integer part of $M \in \mathbb{R}$.

So the $K_\delta(x) = \delta^{-1/2} e^{-\frac{\pi x^2}{\delta}}$ are good kernels.

↳ Helps with approximation theorems.

$f, g \in M(\mathbb{R})$ and let $t \in \mathbb{R}$:

$$(f * g)(t) := \int_{-\infty}^{\infty} f(t-x) g(x)\, dx$$

Proposition:

$f \in S(\mathbb{R})$. Then,

$$\lim_{\delta \to 0} (f * K_\delta)(x) = f(x) \text{ uniformly on } \mathbb{R}.$$

¿ [for $\delta < \delta_0$: $|(f * K_\delta)(x) - f(x)| < \varepsilon \quad \forall\, x \in \mathbb{R}$] ?

$f \in S(\mathbb{R}) \Rightarrow f$ is uniformly continuous on $\mathbb{R}$.

$$\Leftrightarrow \left[\begin{array}{l} \forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : |x-y| < \delta_\varepsilon, \ x, y \in \mathbb{R} : \\ \quad \Rightarrow |f(x) - f(y)| < \varepsilon . \end{array} \right]$$

Proof: f is uniformly continuous.

Let $\varepsilon > 0$:

$$\lim_{x \to \infty} |f(x)| \leq \lim_{x \to \infty} \frac{M}{|x|} = 0$$

$$\Rightarrow \exists M_0, \ |\omega| \geq M_0 \Rightarrow |f(\omega)| < \varepsilon/4 .$$

Since f is u.c. on $[-M_0, M_0]$, then:

$$\exists \delta > 0 : |x-y| < \delta, \quad |x| \leq M_0, \ |y| \leq M_0 \rightarrow |f(x) - f(y)| < \frac{\varepsilon}{4}.$$

$$|x-y| < \delta, \quad |x| \leq M_0, \ |y| \geq M_0$$

$$\Rightarrow |f(x) - f(y)| = |f(x) - f(M_0) + f(M_0) - f(y)|$$

$$\leq |f(x) - f(M_0)| + |f(M_0) - f(y)|$$

$$< \frac{\varepsilon}{4} + |f(M_0)| + |f(y)|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}.$$

$$|x-y| < \delta, \ |x| \geq M_0, \ |y| \geq M_0. \Rightarrow |f(x) - f(y)| < \varepsilon.$$

$$(f * K_\delta)(x) - f(x) = \int_{-\infty}^{\infty} K_\delta(t) f(x-t)\, dt - \int_{-\infty}^{\infty} K_\delta(t) f(x)\, dt$$

$$= \int_{-\infty}^{\infty} K_\delta(t) \{ f(x-t) - f(x) \} dt$$

$$= \int_{|t|\geq\eta} K_\delta(t)\big(f(x-t)-f(x)\big)\,dt + \int_{|t|\leq\eta} K_\delta(t)\big(f(x-t)-f(x)\big)\,dt$$

$$\leq \left| \int_{|t|\geq\eta} K_\delta(t)\big(f(x-t)-f(x)\big)\,dt \right| + \left| \int_{|t|\leq\eta} K_\delta(t)\big(f(x-t)-f(x)\big)\,dt \right|$$

$|f(x)| \leq B$

$$\leq 2B \int_{|t|\geq\eta} K_\delta(t)\,dt + \int_{|t|\leq\eta}$$

UNFINISED. CONTINUE NEXT CLASS.

Let $\varepsilon>0$, $\exists \eta_0$: $|\delta| \leq \eta_0 \Rightarrow \int_{|x|\geq\eta_0} K_\delta(x)\,dx < \varepsilon$