(09.03.21)

CLASS 25

$f, g \in S(\mathbb{R}), \quad x \in \mathbb{R}$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

$$H_x(t) = f(x-t) f(t)$$

Let $\delta > 0$:

$$K_\delta(x) = \delta^{-1/2} e^{-\pi \frac{x^2}{\delta}} \Rightarrow \hat{K}_\delta(\xi) = e^{-\pi \delta \xi^2}.$$

For $\delta = 1$: $K_1(x) = e^{-\pi x^2} = \hat{K}_1(x)$.

i) $\displaystyle\int_{-\infty}^{\infty} K_\delta(x)\, dx = 1 \qquad \forall \delta > 0$

ii) $\displaystyle\int_{-\infty}^{\infty} |K_\delta(x)|\, dx \leq M, \quad \forall \delta > 0$

iii) Fix $\eta > 0$: Then,

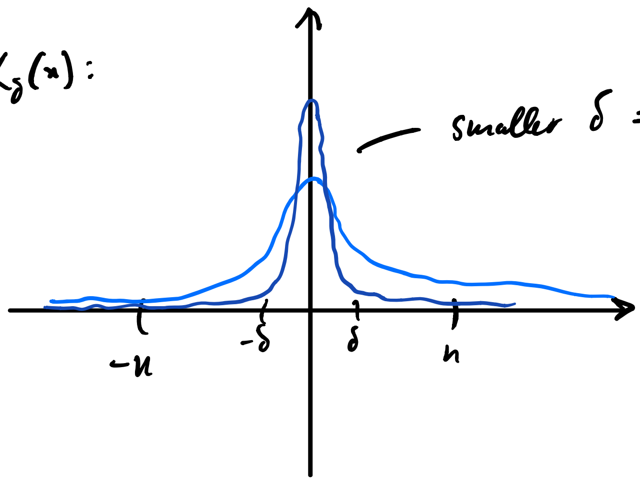
$$\lim_{\delta \to 0} \int_{|x| \geq \eta} |K_\delta(x)|\, dx = 0$$

$\Rightarrow \{K_\delta\}_{\delta > 0}$ is a family of good Kernels.

$f \in S(\mathbb{R})$, then

$$(f * K_\delta)(x) \longrightarrow f(x) \text{ uniformly on } \mathbb{R}.$$

$K_\delta(x)$:



smaller $\delta \Rightarrow K_\delta$ gets more concentrated at the origin. "total mass" = 1.

$$\text{"} \lim_{\delta \to 0} K_\delta(x) = \begin{cases} \text{TOTAL MASS}, & x = 0 \\ 0, & x \neq 0 \end{cases} \text{"}$$

$$\int_{\mathbb{R}} f(x-t) K_\delta(t)\, dt \underset{\delta \to 0}{=} f(x)$$

Let $\varepsilon>0$, $x\in\mathbb{R}$. $0<\delta\leq 1$.

$$|f*K_\delta(x)-f(x)| = \left|\int_{-\infty}^{\infty} f(x-t)K_\delta(t)\,dt - \int_{-\infty}^{\infty} f(x)K_\delta(t)\,dt\right|$$

CLASSIC PART: this difference, then modulus

$$\leq \int_{-\infty}^{\infty} |f(x-t)-f(x)|\,K_\delta(t)\,dt \qquad (*)$$

We know:

i) f is uniformly continuous on $\mathbb{R}$. [f cont. $\Rightarrow$ f unif. cont.]

ii) And $\exists B>0$, $|f(x)|\leq B$ $\forall x\in\mathbb{R}$.

$\exists \eta_0=\eta_0(\varepsilon)$ s.t. for all $|\eta|\leq\eta_0$:

$$|z-w|<\eta \Rightarrow |f(z)-f(w)|<\varepsilon, \qquad z,w\in\mathbb{R}.$$

$$\left.\begin{aligned}&\exists\, \delta_0>0 \text{ s.t. } 0<\delta<\delta_0: \\ &\delta_0=\delta_0(\varepsilon) \\ &\Rightarrow \int_{|x|\geq\eta_0} K_\delta(x)\,dx<\varepsilon\end{aligned}\right\}\text{ by iii)}$$

Split integral $(*)$:

$$\int_{|t|\geq\eta_0} |f(x-t)-f(x)|\,K_\delta(t)\,dt + \int_{|t|\leq\eta_0} |f(x-t)-f(x)|\,K_\delta(t)\,dt$$

$$\leq \underbrace{2B\int_{|t|\geq\eta_0} K_\delta(t)\,dt}_{\text{by ii).}} + \underbrace{\int_{|t|<\eta_0} \varepsilon\, K_\delta(t)\,dt}_{\text{by i).}}$$

$$\leq 2B\varepsilon + \varepsilon \int_{-\infty}^{\infty} K_\delta(t)\,dt$$

(by i).

$$\leq 2B\varepsilon + \varepsilon \cdot 1$$

$$= (2B+1)\varepsilon.$$

So $f * K_\delta(x) \longrightarrow f(x)$ uniformly on $\mathbb{R}$. ∎

PROPERTY: "FUBINI'S THEOREM" → See appendix SS.
→ Exercise!

$$\begin{cases} \text{i) } f \text{ is cont. on } \mathbb{R} \\ \text{ii) } |f(x)| \leq \dfrac{M}{|x|^{1+\delta}}, \quad \delta > 0, \quad x \neq 0 \end{cases}$$

$$\begin{cases} \text{iii) } f \text{ is cont on } \mathbb{R} \\ \text{iv) } |f(x)| \leq \dfrac{A}{1+|x|^{1+\delta}} \quad \forall x \in \mathbb{R}, \ \delta > 0 \end{cases}$$ ← Used in book SS.

i) and ii) is equivalent to iii) and iv).

Proposition: Let $F(x,y)$ continuous and suppose that:

$$|F(x,y)| \leq \frac{A}{(1+|x|^{1+\delta})(1+|y|^{1+\delta})}, \quad (x,y) \in \mathbb{R}^2$$

for some $A>0$, $\delta>0$. Then

i) Define: $$F_1(x) = \int_{-\infty}^{\infty} F(x,y)\,dy.$$

F_1 is a MODERATELY DECREASING FUNCTION.

ii) Define: $$F_2(y) = \int_{-\infty}^{\infty} F(x,y)\,dx.$$

F_2 is a MODERATELY DECREASING FUNCTION.

iii) $$\int_{-\infty}^{\infty} F_1(x)\,dx = \int_{-\infty}^{\infty} F_2(y)\,dy$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(x,y)\,dy\right)dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(x,y)\,dx\right)dy.$$

If you want to change order of integrals, you need some kind of decay.

EXERCISE (APPENDIX.)

BREAK

MULTIPLICATION FORMULA:

Let $f, g \in S(\mathbb{R})$.

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx.$$

<u>Proof:</u> Let $F: \mathbb{R}^2 \to \mathbb{C}$ s.t. $F(x,y) = f(x)g(y)e^{-2\pi i x y}$.

$$|F(x,y)| \leq \frac{A}{1+|x|^{1+\delta}} \cdot \frac{B}{1+|y|^{1+\delta}} = \frac{C}{(1+|x|^{1+\delta})(1+|y|^{1+\delta})}$$

So:

$$F_1(x) = \int_{-\infty}^{\infty} f(x)g(y)e^{-2\pi i x y}\,dy$$

i)
$$= f(x)\int_{-\infty}^{\infty} g(y)e^{-2\pi i x y}\,dy$$

$$= f(x)\,\hat{g}(x).$$

And: $F_2(y) = \int_{-\infty}^{\infty} f(x)g(y)e^{-2\pi i x y}\,dy$

ii)
$$= g(y)\hat{f}(y)$$

iii)
$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)\,dy \qquad \square$$

Fourier Inversion

Lemma: $f \in S(\mathbb{R})$.

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i x t} \, dt$$

$$\Rightarrow \quad \hat{f}(0) = \int_{-\infty}^{\infty} f(t) \, dt$$

! And: $f(0) = \int_{-\infty}^{\infty} \hat{f}(t) \, dt$

↳ **Proof:** $K_\delta(x) = \delta^{-1/2} e^{-\pi \frac{x^2}{\delta}}$

$$\Rightarrow \quad \hat{K}_\delta(t) = e^{-\pi \delta t^2}.$$

$$G_\delta(t) = e^{-\pi \delta t^2} \longrightarrow \hat{G}_\delta = ?$$

$$G_\delta(t) = \frac{(\delta^{-1})^{-1/2}}{(\delta^{-1})^{1/2}} e^{-\pi \frac{t^2}{\delta^{-1}}}$$

? ?

[POOR VIDEO QUALITY]

$\widehat{c \cdot f} = c \hat{f}$, $c \in \mathbb{C}$

$$G_\delta(t) = \frac{1}{(\delta^{-1})^{1/2}} \cdot K_{\delta^{-1}}(t)$$

$$\Rightarrow \hat{G}_\delta(x) = \delta^{-1/2} \hat{K}_\delta(x) = \delta^{-1/2} e^{-\pi \frac{x^2}{\delta}} = K_\delta(x).$$

$$K_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}.$$

$$G_\delta(x) = e^{-\pi \delta x^2}$$

$$\hat{K}_\delta(t) = G_\delta(t)$$

$$\left.\hat{G}_\delta(t) = K_\delta(t)\right\} \quad \widehat{\hat{K}_\delta} = K_\delta$$