(11.03.21)

## CLASS 26:

$\delta > 0$: $\quad K_\delta(x) = \delta^{-1/2} e^{-\pi \frac{x^2}{\delta}} \Rightarrow \hat{K}_\delta(t) = e^{-\pi\delta t^2}$

$$G_\delta(x) = e^{-\pi\delta x^2}$$

$$\hat{K}_\delta = G_\delta \quad \text{and} \quad \hat{G}_\delta = K_\delta.$$

*) $f \in M(\mathbb{R})$, $f \in C^1(\mathbb{R})$, $f' \in M(\mathbb{R})$, then

$$\widehat{f'} = 2\pi i \xi \hat{f} \qquad (*)$$

In particular, (*) holds for Schwartz functions. $(f \in S(\mathbb{R}))$

$f, g \in M(\mathbb{R})$:

$$f * g(x) = \int_{-\infty}^{\infty} f(x-t) g(t)\, dt$$

**Prop:** $f \in M(\mathbb{R})$: Then $(f * K_\delta)(x) \xrightarrow[\delta \to 0]{} f(x)$ uniformly on $\mathbb{R}$.

[Don't need $f \in S$ for this! Even though book assumes $f \in S$.]

**Prop:** $f, g \in M(\mathbb{R}) \Rightarrow \displaystyle\int_{-\infty}^{\infty} f(x)\hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(x) g(x)\, dx$

$$F(x,y) = \frac{A}{(1+|x|^{1+\delta})(1+|y|^{1+\delta})}$$

E.x. $f(x) = \frac{1}{1+x^2}$ is moderately decreasing, but not Schwartz.

can still use prop above!

Ex. favourite function!

$$f(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \leq \frac{1}{\pi^2 x^2}$$

$$\operatorname{sinc} x := \frac{\sin \pi x}{\pi x}$$

Theorem:

If: $f \in M(\mathbb{R})$ and $\hat{f} \in M(\mathbb{R})$

$$(*) \qquad f(t) = \int_{-\infty}^{\infty} \hat{f}(x) e^{2\pi i x t} dx \qquad \text{FOURIER INVERSION}$$

In particular this works for Schwartz functions. [ ¿ Since $f \in S(\mathbb{R}) \Rightarrow \hat{f} \in S(\mathbb{R})$ ]

Proof: $f \in M(\mathbb{R})$ and $\hat{f} \in M(\mathbb{R})$:

$$\Rightarrow f(0) = \int_{-\infty}^{\infty} \hat{f}(x) dx \ ?$$

$\hat{K_\delta} = G_\delta$ and $\hat{G_\delta} = K_\delta$

$$\int_{-\infty}^{\infty} f(x) K_\delta(x)\,dx = \int_{-\infty}^{\infty} f(x) \hat{G_\delta}(x)\,dx$$

Multiplication formula.

$$= \int_{-\infty}^{\infty} \hat{f}(x) G_\delta(x)\,dx$$

$$\underline{\underline{(f*g)(t)}} = \int_{-\infty}^{\infty} f(t-x) g(x)\,dx$$

$[\omega := t-x \text{ and } -dx = d\omega]$

$$= \int_{-\infty}^{\infty} f(\omega) g(t-\omega)\,d\omega$$

CONVOLUTION IS SYMMETRIC

$$= \underline{\underline{(g*f)(t)}}$$

$K_\delta$ is even.

$$\int_{-\infty}^{\infty} f(x) K_\delta(x)\,dx = \int_{-\infty}^{\infty} f(x) K_\delta(-x)\,dx$$

$$= \int_{-\infty}^{\infty} K_\delta(0-x) f(x)\,dx$$

$$= (K_\delta * f)(0) = \int_{-\infty}^{\infty} \hat{f}(x) G_\delta(x)\,dx. \quad (**)$$

LHS in $(**)$ when $\delta \to 0$: $(K_\delta * f)(0) \longrightarrow f(0)$

RHS in $(**)$:

$$\lim_{\delta\to 0} \int_{-\infty}^{\infty} \hat{f}(x) e^{-\pi\delta x^2}\,dx = \int_{-\infty}^{\infty} \hat{f}(x)\,dx$$

WE NEED TO <u>JUSTIFY</u>!

$$\left| \int_{-\infty}^{\infty} \hat{f}(x) e^{-\pi\delta x^2} dx - \int_{-\infty}^{\infty} \hat{f}(x)\, dx \right| = \left| \int_{-\infty}^{\infty} \hat{f}(x) \left( e^{-\pi\delta x^2} - 1 \right) dx \right|$$

$$\leq \int_{-\infty}^{\infty} |\hat{f}(x)| \, |e^{-\pi\delta x^2} - 1| \, dx$$

Have assumed $\hat{f} \in M(\mathbb{R})$ (for some $\eta$)

$$\leq \int_{-\infty}^{\infty} \frac{A}{1+|x|^{1+\eta}} |e^{-\pi\delta x^2} - 1| \, dx$$

See proof below

$0 < \eta < 1$:

$\frac{\eta}{4} < 1$:

$$\leq \int_{-\infty}^{\infty} \frac{A}{1+|x|^{1+\eta}} C (\pi\delta x^2)^{\eta/4} dx$$

$$= A C \pi^{\eta/4} \delta^{\eta/4} \int_{-\infty}^{\infty} \frac{|x|^{\eta/2}}{1+|x|^{1+\eta}} dx$$

$$= K \delta^{\eta/4} \int_{-\infty}^{\infty} \frac{|x|^{\eta/2}}{1+|x|^{1+\eta}}$$

$$\leq K \delta^{\eta/4} \left( \int_{|x|\leq 1} B\, dx + \int_{|x|\geq 1} \frac{1}{|x|^{1+\eta/2}} dx \right)$$

$$= K\delta^{\eta/4} (2B + \text{some number})$$

$$\leq \tilde{K} \delta^{\eta/4} \quad (*)$$

$$\frac{1}{1+|x|^{1+\eta}} \leq \frac{C}{1+|x|^{1+\delta'}}, \quad \text{when } \boxed{\delta' < 1}$$

$\eta < 1 \Rightarrow$ we are done

$\eta \geq 1 \Rightarrow$ E.g. $\eta = 2$: $\frac{1}{1+|x|^3} \leq \frac{C}{1+|x|^{3/2}} \Leftrightarrow \frac{1+|x|^{3/2}}{1+|x|^3} \leq C.$

cont. & $\to 0$ as $|x| \to \infty$ so it's bounded.

So: $$\frac{1}{1+|x|^{1+\eta}} \leq \frac{C}{1+|x|^{3/2}}.$$

[So can pick $\delta < 1$ when $|f(x)| \leq \frac{A}{1+|x|^{1+\delta}}$. But need bigger $A$.]

Let $0<\varepsilon<1 \Rightarrow |e^{-y}-1| \leq c \cdot y^{\varepsilon}$ , $y \geq 0$ , $c=c_{\varepsilon}$.

$y \geq 0$: $|e^{-y}-1| = 1-e^{-y}$

$y \geq 0$: $e^{y} \geq 1 \Rightarrow e^{-y} \leq 1$

L'Hopital

$$\lim_{y\to 0^+} \frac{1-e^{-y}}{y^{\varepsilon}} = \lim_{y\to 0^+} \frac{e^{-y}}{\varepsilon y^{\varepsilon-1}} = \lim_{y\to 0^+} \frac{y^{1-\varepsilon}}{\varepsilon e^{y}} = 0.$$

So: $0 \leq y \leq \delta \Rightarrow \left|\frac{1-e^{-y}}{y^{\varepsilon}}\right| \leq M_{\varepsilon}$

$$|1-e^{-y}| \leq M_{\varepsilon} |y|^{\varepsilon}$$

Now: $$\lim_{y\to\infty} \frac{1-e^{-y}}{y^{\varepsilon}} = \frac{1}{\infty} = 0$$

$$\Rightarrow y \geq N \Rightarrow \frac{|1-e^{-y}|}{|y|^{\varepsilon}} \leq M_{\varepsilon}$$

$$|1-e^{-y}| \leq M_{\varepsilon} |y|^{\varepsilon}$$

$\Rightarrow$ $|1-e^{-y}| \leq F_{\varepsilon} |y|^{\varepsilon}$ on $0 \leq y \leq \delta$ , $y \geq N$.

But what about $0 \le y \le N$?

$$\frac{1-e^{-y}}{y^{\varepsilon}} \to \text{is cont. on compact interval}$$
$$\Rightarrow \text{bounded}$$

$$|1-e^{-y}| \le T_\varepsilon |y|^\varepsilon \quad \text{on } \delta \le y \le N.$$

So $\quad |1-e^{-y}| \le C|y|^\varepsilon$ for all $\underline{y \ge 0}$.

BREAK

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$(*) \implies \delta \to 0: \ f(0) = \int_{-\infty}^{\infty} \hat{f}(x)\,dx.$

So: $g \in M(\mathbb{R}),\ \hat{g} \in M(\mathbb{R}) \Rightarrow g(0) = \int_{-\infty}^{\infty} \hat{g}(x)\,dx.$

Let $f \in M(\mathbb{R})$, $\hat{f}(x) \in M(\mathbb{R})$, $t \in \mathbb{R}$.

Define $g(x) := f(x+t)$.

Then $g \in M(\mathbb{R})$, and $\hat{g}(\xi) = \hat{f}(x)\, e^{2\pi i x t}$. So $\hat{g} \in M(\mathbb{R})$.

$$\begin{array}{ccc} g(0) & = & \displaystyle\int_{-\infty}^{\infty} \hat{g}(x)\,dx \\ \| & & \| \end{array}$$

$$\boxed{f(t) = \int_{-\infty}^{\infty} \hat{f}(x)\, e^{2\pi i x t}\,dx.}$$ For any $t \in \mathbb{R}$

FOURIER INVERSION FORMULA.

$S(\mathbb{R})$:

$\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$.

$f \mapsto \mathcal{F}(t) = \hat{f}$

$$(\mathcal{F}(f))(t) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x t}\,dx.$$

$\mathcal{F}^* : S(\mathbb{R}) \to S(\mathbb{R})$

$f \mapsto \mathcal{F}^*(f)$ , $\qquad (\mathcal{F}^*(f))(t) = \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{2\pi i x t}\,dx$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x(-t)}\,dx$$

$$\boxed{f(t) = \int_{-\infty}^{\infty} \hat{f}(x) e^{2\pi i x t}\,dx} = \mathcal{F}^*(\hat{f})(t) = f(t)$$

so: $f = \mathcal{F}^*(\hat{f})$

$$f = \mathcal{F}^*(\mathcal{F}(f))$$

So: $\mathcal{F}^* \circ \mathcal{F} = Id.$

composition identity operator

¿ Also: $\mathcal{F} \circ \mathcal{F}^* = Id$ ?

$$\left(\mathcal{F}(\mathcal{F}^*)\right)(f)(t) = f(t)?$$

$$\left(F^*(f)\right)(t) = \mathcal{F}(f)(-t)$$

$$\mathcal{F}(\mathcal{F}^*(f))\, t = \int_{-\infty}^{\infty} (\mathcal{F}^*(f))(x)\, e^{-2\pi i x t}\,dx$$

$$= \int_{-\infty}^{\infty} \mathcal{F}(f)(-x) e^{-2\pi i x t}\,dx \quad , \quad -x = \omega$$

Change of variables.

$$= \int_{-\infty}^{\infty} \mathcal{F}(f)(\omega) e^{2\pi i \omega t}\,d\omega$$

$$= \int_{-\infty}^{\infty} \hat{f}(\omega) e^{2\pi i \omega t}\,d\omega$$

Have proved.

$$= f(t)$$

$$\Rightarrow \quad \mathcal{F} \circ \mathcal{F}^* = Id.$$
$$\mathcal{F}^* \circ \mathcal{F} = Id.$$

$\Rightarrow$ $\mathcal{F}$ is a BIJECTIVE OPERATOR on $S(\mathbb{R})$ !

injective

surjective "onto"

$$\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$$

$$g \in S(\mathbb{R}) \Rightarrow \exists f \in S(\mathbb{R}) \text{ s.t. } g = \hat{f}.$$

$$\mathcal{F}(\mathcal{F}^*) = Id, \quad \mathcal{F}^*(\mathcal{F}) = Id$$

Exercise: $\mathcal{F}(\mathcal{F}) = R$, where $R(f)(x) = f(-x)$.

Reflection.

$$\widehat{(\hat{f})}(x) = f(-x).$$

$$\hat{K}_\delta = G_\delta \text{ and } \hat{G}_\delta = K_\delta$$

$$\hat{\hat{K}}_\delta = \hat{G}_\delta$$
$$\hat{\hat{K}}_\delta(x) = K_\delta(-x).$$

$$\mathfrak{F} \circ \mathfrak{F} \circ \mathfrak{F} \circ \mathfrak{F} = Id.$$

!

(for any $f \in \mathcal{M}(\mathbb{R})$, $\hat{f} \in \mathcal{M}(\mathbb{R})$ ?)