(16.03.21)

Class 28

News: * Problem set 4.

Fourier Inversion

$f, \hat{f} \in M(\mathbb{R})$. $\qquad f, \hat{f} \in L^1(\mathbb{R})$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i t x} \, dt \quad \rightsquigarrow \quad \text{a.e.}$$

Proposition:

i) $f, g \in S(\mathbb{R}) \Rightarrow f * g \in S(\mathbb{R})$

ii) $f, g \in M(\mathbb{R}) \Rightarrow f * g \in M(\mathbb{R})$ $\leftarrow$ Exercise!

Proof of i):

a) $\sup_{x \in \mathbb{R}} |x|^k (f*g)(x) < \infty$ for any $k \geq 0$.

(We need to prove this.)

$$|x|^k (f*g)(x) = |x|^k \int_{-\infty}^{\infty} f(x-y) g(y) \, dy$$

$$= \int_{-\infty}^{\infty} |x|^k f(x-y) g(y) \, dy$$

$$|x|^k |f(x-y)| \le A(1+|y|)^k \quad \forall x \in \mathbb{R} \quad \forall y \in \mathbb{R}$$

*) $|x| \le 2|y|$:

[$f \in S \Rightarrow \exists B$ s.t. $|f| \le B$]

$$\Rightarrow |x|^k |f(x-y)| \le B|x|^k \le B(2|y|)^k$$
$$= B2^k (|y|)^k$$
$$\le B \cdot 2^k (1+|y|)^k$$

*) $|x| > 2|y|$:

Δ-ineq:

$$|x-y| + |y| \ge |x|$$

$$|x-y| \ge |x| - |y|$$
$$\ge \frac{|x|}{2}$$

$f \in S \Rightarrow |z|^k |f(z)| \le B_1$

$f(z) \le \frac{B_1}{|z|^k}$ $z \neq 0$.

$$\Rightarrow |x|^k |f(x-y)| \le |x|^k \frac{B_1}{|x-y|^k}$$
$$\le |x|^k B_1 \left(\frac{2}{|x|}\right)^k$$
$$= B_1 \cdot 2^k$$
$$\le B_1 \cdot 2^k (1+|x|)^k.$$

$$\left| |x|^k (f*g)(x) \right| \le \int_{-\infty}^{\infty} |x|^k |f(x-y)| \, |g(y)| \, dy$$
$$\le \int_{-\infty}^{\infty} A(1+|y|)^k |g(y)| \, dy$$
$$\le \int_{-\infty}^{\infty} A \, (1+|y|)^k \frac{c}{(1+|y|)^{k+2}} \, dy$$

$$\leq AC \underbrace{\int_{-\infty}^{\infty} \frac{1}{(1+|y|)^2}\, dy}_{\text{this is a finite number.}} .$$

So: $\sup_{x \in \mathbb{R}} \left| |x|^k (f * g)(x) \right| < \infty .$

b) Assume $g : \mathbb{R} \to \mathbb{R}$. (If $g : \mathbb{R} \to \mathbb{C}$ then $g = \operatorname{Re} g + \operatorname{Im} g$.)

$$(f*g)'(x) = (f*g')(x) = (f'*g)(x)$$

Fix $x \in \mathbb{R}$:

$$\lim_{h \to 0} \underbrace{\left(\frac{(f*g)(x+h) - (f*g)(x)}{h} - (f*g')(x) \right)}_{K_h} = 0$$

$$K_h = \int_{-\infty}^{\infty} f(y) \left(\frac{g(x+h-y) - g(x-y)}{h} - g'(x-y) \right) dy$$

Trick not in book:

$h > 0$: $$\frac{g(x+h-y) - g(x-y)}{h} - g'(x-y) = \frac{\int_{x-y}^{x-y+h} g'(t)\, dt}{h} - \frac{\int_{x-y}^{x-y+h} g'(x-y)\, dt}{h}$$

$$= \frac{\int_{x-y}^{x-y+h} (g'(t) - g'(x-y))\, dt}{h} \quad (\alpha)$$

$g: \mathbb{R} \to \mathbb{R}$. Mean Value Theorem:

$t \in [x-y, x-y+h]$: $\quad g'(t) - g'(x-y) = g''(\xi)(t-(x-y)), \quad \xi \in [x-y, t]$

$$|g'(t) - g'(x-y)| \le |g''(\xi)|\,|t-(x-y)|$$
$$\le B_2\,|t-(x-y)|$$

$$x-y \le t \le x-y+h]$$

$$0 \le t-(x-y) \le h$$

$$|g'(t) - g'(x-y)| \le B_2 h \quad (\beta)$$

Using (α) and (β) in K_h:

$$|K_h| \le \int_{-\infty}^{\infty} |f(y)| \cdot \left(\int_{x-y}^{x-y+h} \frac{|g'(t) - g'(x-y)|}{h}\, dt \right) dy \quad (\leftarrow \text{by } \alpha.)$$

$$\le \int_{-\infty}^{\infty} |f(y)| \left(\int_{x-y}^{x-y+h} \frac{B_2 h}{h}\, dt \right) dy$$

$$= \int_{-\infty}^{\infty} |f(y)|\, dy\, B_2 h. \qquad \text{So: } |K_h| \le \int_{-\infty}^{\infty} |f(y)|\, dy\, B_2 h$$

$$|K_h| \to 0 \text{ as } h \to 0. \blacksquare$$

OBSERVATION: $f, g \in S(\mathbb{R})$

$$\sup_{x \in \mathbb{R}} |x|^k (f*g)(x) < \infty$$

$$(f*g)'(x) = (f*g')(x)$$

$k, \ell \geq 0$: $$\sup_{x\in\mathbb{R}} |x|^k (f*g)^{(\ell)}(x) = \sup_{x\in\mathbb{R}} |x|^k (f * g^{(\ell)})(x)$$

$$< \infty \quad b/c \quad g \in S(\mathbb{R}) \Rightarrow g^{(\ell)} \in S(\mathbb{R}).$$

i) $f*g \in S(\mathbb{R})$ if $f,g \in S(\mathbb{R})$.

ii) $f*g \in M(\mathbb{R})$ if $f,g \in M(\mathbb{R})$

iii) $\widehat{f*g}(\xi) \underset{(*)}{=} \hat{f}(\xi)\hat{g}(\xi)$ when $f,g \in S(\mathbb{R})$.

iv) $(*)$ Holds also for $f,g \in M(\mathbb{R})$. Assume iv) is OK.
Takes 1 class to prove, using L-measure.

<u>Proof of iii)</u> Let $\xi \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$.

$$F(x,y) = f(y)g(x-y)e^{-2\pi i \xi x}.$$

$$|F(x,y)| \leq \frac{M}{(1+|x|^2)(1+|y|^2)}. \quad (**)$$ ← We want to prove this.

Proof of (**):

$|x|^k |f(x)| < \infty \Rightarrow$

$\sup_{x\in\mathbb{R}} |P(x) f(x)| < \infty$

so: $|P(x) f(x)| \le M$

$|f(x)| \le \frac{M}{P(x)}$.

·) $|x| \geq 2|y|$:

$$|F(x,y)| = |f(y)|\,|g(x-y)| \le \frac{A}{1+|y|^2} \cdot \frac{B}{1+|x-y|^2}$$

$$|x-y|^2 \overset{\Delta\text{ ineq}}{\geq} |x|-|y| \overset{\text{assumption}}{\geq} \frac{|x|}{2}$$

$$\le \frac{A}{1+|y|^2} \cdot \frac{B}{1+\left(\frac{|x|}{2}\right)^2}$$

$$\le \frac{AB}{1+|y|^2} \cdot \frac{4}{4+|x|^2} \le \frac{4AB}{(1+|y|^2)(1+|x|^2)}.$$

·) $|x| < 2|y|$:

b/c g is bounded.

$$|F(x,y)| \le |f(y)| \cdot C$$

$f \in S(\mathbb{R})$.

$$\le \frac{N \cdot C}{(1+|y|^2)^2}$$

$|x| \le 2|y|$

$\Rightarrow \frac{|x|^2}{4} < |y|^2$

$$= \frac{NC}{(1+|y|^2)(1+|y|^2)}$$

$$= \frac{NC}{(1+|y|^2)\left(1+\frac{|x|^2}{4}\right)}$$

$$\frac{1}{1+\frac{|x|^2}{4}} = \frac{1}{\frac{4}{4}+\frac{|x|^2}{4}} = \frac{4}{4+|x|^2} \le \frac{4}{1+|x|^2}$$

$$\Rightarrow |F(x,y)| \le \frac{NC \cdot 4}{(1+|y|^2)(1+|x|^2)}.$$

So (**) works for any x, y.

$$F(x,y) = f(x)\,g(x-y)\,e^{-2\pi i \xi x}.$$

Using "FUBINI'S THM" / CHANGE OF ORDER OF INTEGRATION.

$$F_1(x) = \int_{-\infty}^{\infty} F(x,y)\,dy = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,e^{-2\pi i \xi x}\,dy$$

$$= e^{-2\pi i \xi x} \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy$$

$$= e^{-2\pi i \xi x} (f * g)(x).$$

$$F_2(y) = \int_{-\infty}^{\infty} F(x,y)\,dx = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,e^{-2\pi i x \xi}\,dx$$

$$= f(y) \int_{-\infty}^{\infty} g(x-y)\,e^{-2\pi i x \xi}\,dx$$

change of var's: $\omega = x - y$

$$= f(y) \int_{-\infty}^{\infty} g(\omega)\,e^{-2\pi i (\omega + y)\xi}\,d\omega$$

$$= f(y)\,e^{-2\pi i y \xi} \int_{-\infty}^{\infty} g(\omega)\,e^{-2\pi i \omega \xi}\,d\omega$$

$$= f(y)\,e^{-2\pi i y \xi}\,\hat{g}(\xi).$$

$$\int_{-\infty}^{\infty} F_1(x)\,dx = \int_{-\infty}^{\infty} F_2(y)\,dy$$

$$\int_{-\infty}^{\infty} e^{-2\pi i x \xi}(f*g)(x)\,dx = \int_{-\infty}^{\infty} f(y) e^{-2\pi i \xi y}\,\hat{g}(\xi)\,dy$$

$$\widehat{(f*g)}(\xi) = \hat{g}(\xi)\cdot\hat{f}(\xi).$$

PLANCHEREL:

Let $f \in \mathcal{M}(\mathbb{R})$, $\hat{f} \in \mathcal{M}(\mathbb{R})$.

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2\,dx$$

PROOF:

DEF: $g(x) := \overline{f(-x)}$. $\Rightarrow g \in \mathcal{M}(\mathbb{R})$.

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i x \xi}\,dx = \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{-2\pi i \xi x}\,dx$$

$\downarrow -x = \omega$

$$= \int_{-\infty}^{\infty} \overline{f(\omega)}\, e^{-2\pi i (-\omega)}\,d\omega$$

$$= \int_{-\infty}^{\infty} \overline{f(\omega)}\, e^{2\pi i \omega \xi}\, d\omega$$

$$= \int_{-\infty}^{\infty} \overline{f(\omega)\, e^{-2\pi i \omega \xi}}\, d\omega$$

$$= \overline{\int_{-\infty}^{\infty} f(\omega)\, e^{-2\pi i \omega \xi}\, d\omega}$$

$$= \overline{\hat{f}(\xi)}\,.$$

Def: $h := f * g$, $\Rightarrow h \in M(\mathbb{R})$ b/c $f, g \in M(\mathbb{R})$.

$$\hat{h} = \widehat{f * g} = \hat{f} \cdot \hat{g} = \hat{f}(\xi) \cdot \overline{\hat{f}(\xi)} = |\hat{f}(\xi)|^2$$

$\downarrow$ $\hat{g} = \overline{\hat{f}(\xi)} \in M(\mathbb{R})$.

So $\hat{h} \in M(\mathbb{R})$.

So: $h(x) = \int_{-\infty}^{\infty} \hat{h}(t)\, e^{2\pi i x t}\, dt$ $\leftarrow$ FOURIER INVERSION

Let now $x = 0$:

$$h(0) = \int_{-\infty}^{\infty} \hat{h}(t)\, dt.$$

And $h(0) = \int_{-\infty}^{\infty} f(x)\, g(0-x)\, dx = \int_{-\infty}^{\infty} f(x) g(-x)\, dx = \int_{-\infty}^{\infty} f(x)\, \overline{f(x)}\, dx = \int_{-\infty}^{\infty} |f(x)|^2 dx$

So: $h(0)$:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \hat{f}(x)\, \hat{g}(x)\, dx$$

$$= \int_{-\infty}^{\infty} \hat{f}(x)\, \overline{\hat{f}(x)}\, dx$$

$$= \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx \quad \blacksquare$$