(18.03.21)

Class 29: Poisson Summation Formula

Important class!

News: * No exercise class today!

Poisson Summation Formula (PSF):

Let $f \in M(\mathbb{R})$, $\hat{f} \in M(\mathbb{R})$.

Then: $$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x} \quad \forall x \in \mathbb{R}.$$

In particular: $\displaystyle\sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} \hat{f}(k)$.
$(x=0)$

<u>Proof:</u>

$$G_N(x) := \sum_{n=-N}^{N} f(x+n).$$

Let $M \geq 1$: Consider $x \in [-M, M]$, $\underset{?}{\sim\!\sim}$

Take $N \geq 2M+1$.

$$G_N(x) = \sum_{|n| \leq 2M} f(x+n) + \sum_{2M < |n| \leq N} f(x+n).$$ $\underset{?}{\sim\!\sim}$

$$\sum_{2M<|n|\le N} |f(x+n)| \le \sum_{2M<|n|\le N} \frac{C}{1+|x+n|^{1+\delta}} \qquad \text{Since } |f(x)| \le \frac{C}{1+|x|^{1+\delta}}.$$

$x \in [-M, M]$.

Δ-ineq.

$$|x+n| \ge |n| - |x| \ge \frac{|n|}{2}.$$

$$|n| > 2M \ge 2|x|$$

$$|n| > 2|x|$$

$$\le \sum_{2M<|n|\le N} \frac{C}{1+\left(\frac{|n|}{2}\right)^{1+\delta}}$$

$$\le C2^{1+\delta} \underbrace{\sum_{2M<|n|\le N} \frac{1}{|n|^{1+\delta}}}_{\text{This sum converges.}} \quad \left(\text{Integral test: } \int_1^\infty \frac{dt}{t^{1+\delta}}.\right)$$

Test: Weierstrass criterion $\Rightarrow$ $\sum_{2M<|n|<N} f(x+n)$ is abs. & unif. convergent as $N \to \infty$.

$\Rightarrow$ $\lim_{N\to\infty} G_N(x)$ exists and also:

$G(x) = \sum_{n\in\mathbb{Z}} f(x+n)$ is a function absolutely convergent & uniformly cont. on any compact sets.

$$G(x+1) = \sum_{n\in\mathbb{Z}} f(x+n+1) = \sum_{n\in\mathbb{Z}} f(x+n) = G(x).$$

So G is a 1-periodic function.

Also: $G: \mathbb{R} \to \mathbb{C}$ b/c G is def on any compact.

Also: G is continuous b/c G_N converges uniformly.

$$\hat{G}(k) = \int_0^1 G(x) e^{-2\pi i k x}\, dx, \quad k \in \mathbb{Z}.$$

$$= \int_0^1 \left(\sum_{n\in\mathbb{Z}} f(x+n)\right) e^{-2\pi i k x}\, dx$$

b/c we have uniform convergence we can swap sum & integral.

$$= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i k x}\, dx$$

$(x+n = y)$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(y) e^{-2\pi i k (y-n)}\, dy$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(y) e^{-2\pi i k y}\, dy$$

$$= \int_{-\infty}^{\infty} f(y) e^{-2\pi i k y}\, dy.$$

So: $\hat{G}(k) = \hat{f}(k)$, $k \in \mathbb{Z}$.

PERIODIZATION OF f.

$$G(x) = \sum_{n\in\mathbb{Z}} f(x+n).$$

$$H_N(x) := \sum_{|k|\le N} \hat{f}(k) e^{2\pi i k x}.$$

b/c $\hat{f} \in M(\mathbb{R})$.

$$\sum_{|k|\le N} |\hat{f}(k) e^{2\pi i k x}| = \sum_{|k|\le N} |\hat{f}(k)| \le \sum_{|k|\le N} \frac{B}{1+|k|^{1+\delta}}$$

$$= B\left\{ \frac{1}{1} + \sum_{\substack{|k|\le N \\ k\ne 0}} \frac{1}{|k|^{1+\delta}} \right\}.$$

$\Rightarrow$ By Weierstrass Test:

$H(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$ is a continuous function. (b/c convergence is uniform.)

BREAK

$$G(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

G is a cont. 1-periodic function.

$\hat{G}(k) = \hat{f}(k)$, $k \in \mathbb{Z}$

$$H(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$$

H is a cont. <u>1-periodic</u> function.

$$H(x+1) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k (x+1)} = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x} = H(x)$$

$\hat{H}(k) = \hat{f}(k), \quad k \in \mathbb{Z}.$

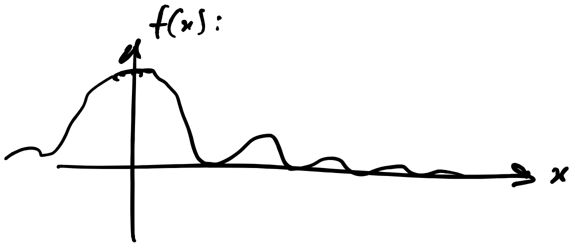
$$\hat{H}(m) = \int_0^1 H(x) e^{-2\pi i k x} dx = \hat{f}(m).$$

Using our <u>First Theorem of Fourier Series</u>:

$$\left[\hat{f}(k) = \hat{g}(k) \ \forall k \Rightarrow f(x) = g(x). \right]$$

$$\Rightarrow \sum_{n \in \mathbb{Z}} f(x+n) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}.$$

$$f(x) = \left(\frac{\sin \pi x}{\pi x} \right)^2 =: \operatorname{sinc}^2(x)$$

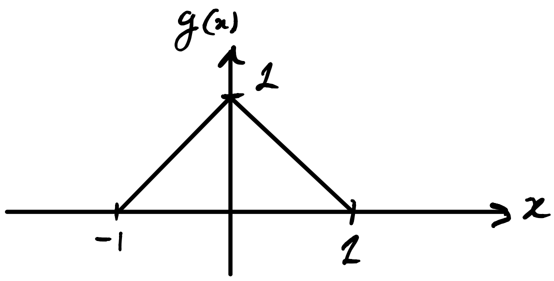


$|x| \geq 1$: $\quad |f(x)| \leq \dfrac{1}{\pi^2 x^2} \quad$ so $f \in M(\mathbb{R})$.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \left(\frac{\sin \pi x}{\pi x} \right)^2 e^{-2\pi i \xi x} dx \longrightarrow$$

Difficult calculation.
Cauchy Integral Thm
Complex Analysis.

$$g(x) = \max\{1 - |x|, 0\}$$



$\xi \neq 0$:

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i \xi x} dx = \int_{-1}^{1} (1-|x|) e^{-2\pi i \xi x} dx$$

$$= \int_{-1}^{0} (1+x) e^{-2\pi i \xi x} dx + \int_{0}^{1} (1-x) e^{-2\pi i \xi x} dx.$$

$\omega = -x$:

$$= \int_{0}^{1} (1-\omega) e^{2\pi i \xi \omega} d\omega + \int_{0}^{1} (1-x) e^{-2\pi i \xi x} dx.$$

$$\int_{0}^{1} (1-\omega) e^{2\pi i \xi \omega} d\omega = \int_{0}^{1} (1-\omega) \left(\frac{e^{2\pi i \xi \omega}}{2\pi i \xi} \right)' d\omega$$

$$= (1-\omega) \frac{e^{2\pi i \xi \omega}}{2\pi i \xi} \Bigg|_{\omega=0}^{\omega=1} - \int_{0}^{1} (-1) \left(\frac{e^{2\pi i \omega}}{2\pi i \xi} \right) d\omega$$

$$= \frac{-1}{2\pi i \xi} + \frac{1}{2\pi i \xi} \int_{0}^{1} e^{2\pi i \omega \xi} d\omega$$

$$= -\frac{1}{2\pi i \xi} + \frac{1}{2\pi i \xi} \left\{ \frac{e^{2\pi i \omega \xi}}{2\pi i \xi} \Bigg|_{\omega=0}^{\omega=1} \right\}$$

$$= -\frac{1}{2\pi i \xi} + \frac{1}{2\pi i \xi} \left\{ \frac{e^{2\pi i \xi}}{2\pi i \xi} - \frac{1}{2\pi i \xi} \right\}.$$

$g(x) = \max\{1-|x|, 0\}$.

$$\hat{g}(x) = \frac{-1}{2\pi i \xi} + \frac{e^{2\pi i \xi} - 1}{(2\pi i \xi)^2} + \left(\frac{-1}{2\pi i(-\xi)} + \frac{e^{-2\pi i \xi} - 1}{(2\pi i(-\xi))^2} \right)$$

$$= \frac{e^{2\pi i \xi} - 1 + e^{-2\pi i \xi} - 1}{(2\pi i \xi)^2}$$

$$= \frac{e^{2\pi i \xi} - 2 + e^{-2\pi i \xi}}{(2\pi i \xi)^2}$$

$$= \frac{(e^{\pi i \xi} - e^{-\pi i \xi})^2}{(2i)^2 (\pi \xi)^2}$$

$$= \left(\frac{e^{\pi i \xi} - e^{-\pi i \xi}}{2i} \right)^2 \frac{1}{(\pi \xi)^2}$$

$$= \frac{\sin^2(\pi \xi)}{(\pi \xi)^2} = \left(\frac{\sin(\pi \xi)}{\pi \xi} \right)^2.$$

$f(x) = \left(\frac{\sin \pi x}{\pi x} \right)^2$, $g(x) = \max\{1 - |x|, 0\}$

$\hat{\hat{g}} = \hat{f}$

$g(-x) = \hat{f}(x)$

g is even. $g(x) = \hat{f}(x)$.

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi x}{\pi x} \right)^2 dx = \hat{f}(0) = g(0) = 1.$$

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}.$$

$\forall x \in \mathbb{R}$: $$\sum_{n\in\mathbb{Z}} \left(\frac{\sin(\pi(x+n))}{\pi(x+n)}\right)^2 = \hat{f}(0) = 1.$$

$\lambda > 0$:

$$P_\lambda(x) = \frac{1}{\pi} \frac{\lambda}{\lambda^2+x^2} \Rightarrow \hat{P}_\lambda(x) = e^{-2\pi\lambda|x|}.$$

$$\frac{1}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{1+(x+n)^2} = \sum_{k\in\mathbb{Z}} e^{-2\pi|k|} e^{2\pi i k x}$$

$$\frac{1}{\pi} \sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \underbrace{\sum_{k\in\mathbb{Z}} e^{-2\pi|k|}}$$

geometric series: $= \dfrac{1+e^{-2\pi}}{1-e^{-2\pi}}.$

$$G(x) = e^{-\pi x^2}$$

$$\sum_{n \in \mathbb{Z}} e^{-\pi(x+n)^2} = \sum_{k \in \mathbb{Z}} e^{-\pi k^2} e^{2\pi i k x} \quad \forall x \in \mathbb{R}.$$