(23.03.21)

$\checkmark$ = CLASS
$\times$ = NO CLASS

CLASS 30

NEWS:

	Tues	Thur		Week number
	$\checkmark$ RZ-func	$\checkmark$ → RZ-func → Heisenberg		12
	$\times$	$\times$	} Easter	13
	$\times$	$\checkmark$ — PLT Paley Wiener Thm		14
April By Zoom ←	$\checkmark$ PLT	$\checkmark$ — PLT Shannon Interpol		15
	$\checkmark$	$\checkmark$ or VI		16
	$\checkmark$ Final CLASS Review	or Wavelets		17

April 6th
No CLASS

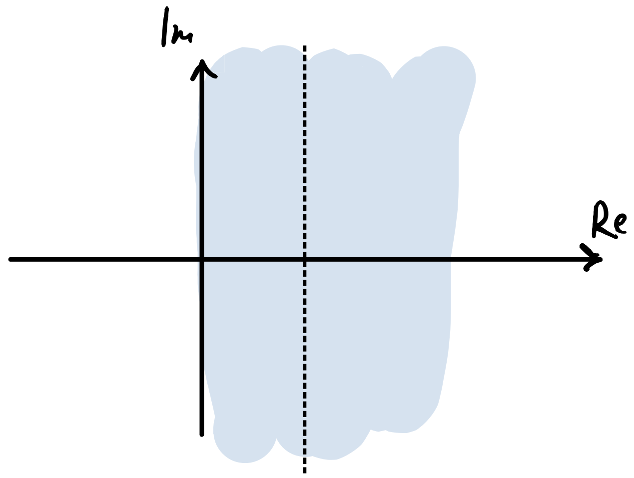
For PLT we follow complex analysis book by Stein & Shakarchi.

THE RIEMANN ZETA-FUNCTION

•) The Gamma Function

$s \in \Omega \subset \mathbb{C}$ Def. Analytic:

$$\lim_{s \to s_0} \frac{f(s) - f(s_0)}{s - s_0} = f'(s_0)$$



$s \in \mathbb{C}$, $\mathrm{Re}\, s > 0$

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt.$$

$\mathrm{Re}\, s > 0$: $s = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$. $\sigma > 0$

$$\int_0^\infty |e^{-t} t^{s-1}| \, dt = \int_0^\infty e^{-t} t^{\sigma-1} \, dt$$

$$= \int_0^1 e^{-t} t^{\sigma-1} \, dt + \int_1^\infty e^{-t} t^{\sigma-1} \, dt$$

$0 < t < 1$
$-1 < -t < 0$
$e^{-1} < e^{-t} < 1$

$$\leq \int_0^1 t^{\sigma-1} \, dt + \underbrace{\int_1^\infty e^{-t} t^{\sigma-1} \, dt}_{\approx \int_1^\infty e^{-t} dt}$$

$$\leq \underbrace{\frac{t^\sigma}{\sigma}\Big|_0^1}_{\frac{1}{\sigma}} + \int_1^\infty C_\sigma e^{-t/2}\,dt$$

$e^{-t}t^{\sigma-1} \leq C_\sigma e^{-t/2} \Leftrightarrow t^{\sigma-1} \leq C_\sigma e^{t/2}$

$\lim_{t\to\infty} \frac{t^{\sigma-1}}{e^{t/2}} = 0$

*) Γ is an analytic function $\operatorname{Re} s > 0$.

(Morera's Theorem: $\oint_\gamma f(z)\,dz = 0 \quad \forall \gamma$

$\Rightarrow f$ analytic.

→ i) f cont. in Ω. Ω open set.

ii) For any triangle $\Delta \subset \Omega$: $\int_{\partial\Delta} f(z)\,dz = 0$.

$\Rightarrow f$ is analytic.)

$f_n : \Omega \to \mathbb{C}$

f_n converges to f uniformly in compacts: $f_n \xrightarrow{\text{uniformly}} f$

$\Rightarrow f_n$ is analytic in Ω.

$$\Gamma_N(s) = \int_{1/n}^{n} e^{-t} t^{s-1}\, dt.$$

$$\left.\begin{aligned} &\Gamma_N \text{ is cont.} \\ &\int_{\partial\Delta} \Gamma_N\, dz = 0 \end{aligned}\right\} \Gamma_N \text{ is anal. (Morera)}$$

$\Gamma_N \to \Gamma$ uniformly, so Γ is analytic.

Lemma: For $\operatorname{Re} s > 0$:

$$\Gamma(s+1) = s\,\Gamma(s). \qquad \Gamma(1) = 1.$$

Proof:

$n \geq 1$:

$$\int_{1/n}^{n} \left(e^{-t} t^{s}\right)' dt = \int_{1/n}^{n} \left\{ e^{-t}(-1)t^{s} + e^{-t} s t^{s-1} \right\} dt$$

$$\underbrace{e^{-n} n^{s}}_{\substack{\to 0 \\ \text{as } n\to\infty}} - \underbrace{e^{-\frac{1}{n}} \left(\tfrac{1}{n}\right)^{s}}_{\substack{\to 0 \\ \text{as } n\to\infty}} = -\int_{1/n}^{n} e^{-t} t^{s+1-1}\, dt + s \int_{1/n}^{n} e^{-t} t^{s-1}\, dt$$

Let $\operatorname{Re} s > 0$ and let $n \to \infty$:

$$0 = -\Gamma(s+1) + s\,\Gamma(s)$$

so: $\Gamma(s+1) = s\,\Gamma(s)$. ∎

- $\Gamma(1) = \int_0^\infty e^{-t}\,dt = \frac{e^{-t}}{-1}\Big|_0^\infty = 1$
- $\Gamma(2) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1$
- $\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2$
- $\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 = 6$

$\vdots$

- $\Gamma(n) = (n-1)! \quad \forall\, n \geq 1.$

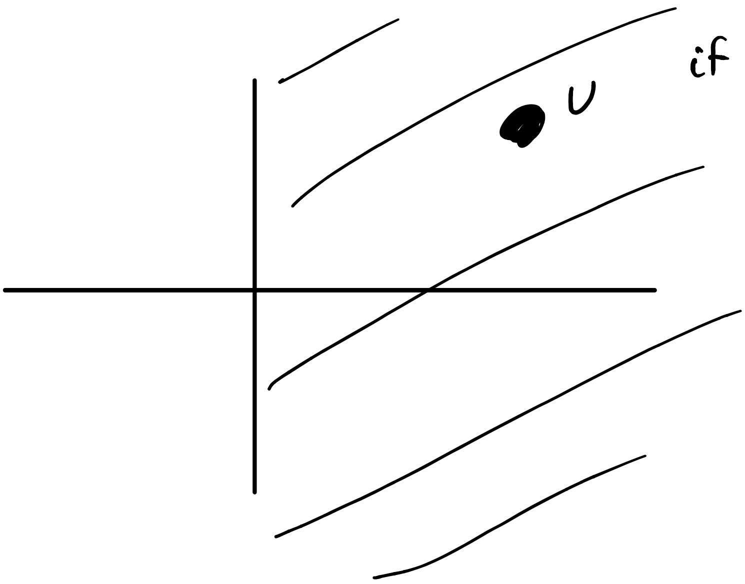
Analytic continuation

$f, g : \Omega \to \mathbb{C}$ analytic

$\Omega \subset \mathbb{C}$ connected open set

$f(s) = g(s)$ in $U \subset \Omega$
$\quad\quad\quad\quad\quad\quad \downarrow$ open set

$\Rightarrow f \equiv g$



if $f = g$ on U

$\Rightarrow f = g$ on all of Ω.

Not true in real analysis! Only complex.

BREAK

Lemma: Γ has an analytic continuation to all of $\mathbb{C}$.

i.e. there is a meromorphic function

has poles. (isolated singularities, not essential sings.)

F defined on $\mathbb{C}$ such that

$$F(s) = \Gamma(s) \text{ in } \mathrm{Re}\, s > 0.$$

Proof: Define: $F_1 : \{s : \mathrm{Re}\, s > -1\} \to \mathbb{C}$

$$F_1(s) = \frac{\Gamma(s+1)}{s}$$

F_1 is meromorphic.

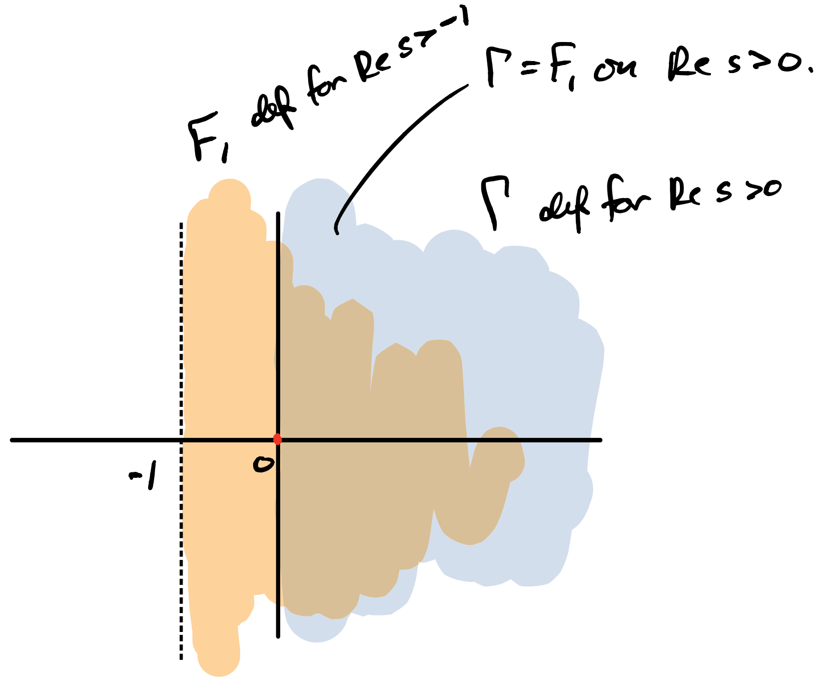
F_1 is a meromorphic function with a pole at $s=0$.

In $\mathrm{Re}\, s > 0$:

$$F_1(s) = \frac{\Gamma(s+1)}{s} \overset{\text{Lemma}}{=} \Gamma(s)$$

So F_1 extends Γ.

Can def F_2, F_3, etc.



Define $F_2 : \{ s \in \mathbb{C} : \operatorname{Re} s > -2 \} \to \mathbb{C}$

$$F_2(s) = \frac{\Gamma(s+2)}{s(s+1)} = \frac{(s+1)\Gamma(s+1)}{s(s+1)} = \frac{(s+1)s\,\Gamma(s)}{s(s+1)} = \Gamma(s)$$

In $\operatorname{Re} s > -2$: $F_2(s) = \dfrac{\Gamma(s+2)}{s+1} \cdot \dfrac{1}{s}$

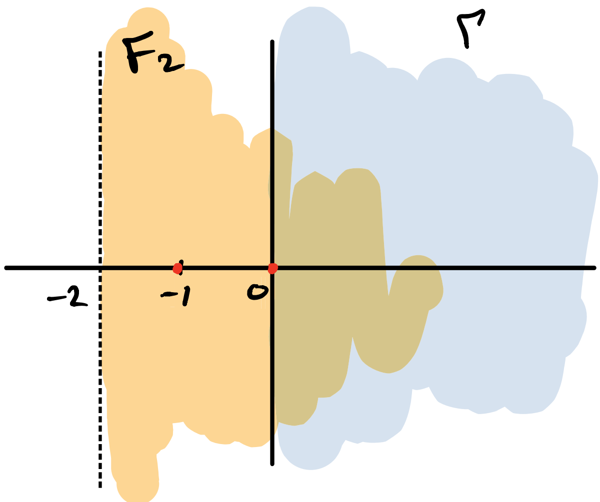
$\operatorname{Re}\{s+1\} > -1$

$$= \underbrace{\frac{\Gamma(s+1+1)}{s+1}}_{=F_1(s+1)} \cdot \frac{1}{s} = \frac{F_1(s+1)}{s}$$

F_2 is meromorphic in $\{ s \in \mathbb{C}, \operatorname{Re} s > -1 \}$ with poles at $s = 0$ and at $s = -1$.

$F_1(1) = \Gamma(2) = 1$

In $\operatorname{Re} s > 0$: $F_2(s) = \dfrac{\Gamma(s+1)}{s} = \Gamma(s)$.



Γ is analytic in $\overbrace{\mathbb{C}\setminus\{0,-1,-2,\dots\}}^{=:U}$.

Define $G: U \to \mathbb{C}$ analytic:

$$G(s) := \Gamma(s+1) - s\Gamma(s)$$

$$G(s) = 0 \text{ in } \operatorname{Re} s > 0.$$

$\Rightarrow$ By identity theorem $G(s) = 0$ in U.

so: $\Gamma(s+1) = s\Gamma(s)$ in U.

$$\Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{\left(\frac{-3}{2}\right)} = \frac{\Gamma\left(\frac{-1}{2}\right)}{\left(\frac{-3}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)} = \frac{4}{3}\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\text{compute using integral.}} = \frac{4}{3}\sqrt{\pi}.$$