Class 32: The Riemann zeta function 1859

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

$x > 1$ $\quad \underbrace{\sum_{n=1}^{N} \frac{1}{n^x}}$

$n-1 \le t \le n$, $t \in [n-1, n]$

$$\frac{1}{n} \le \frac{1}{t} \Rightarrow \frac{1}{n^x} \le \frac{1}{t^x} \Rightarrow \int_{n-1}^{n} \underbrace{\frac{dt}{n^x}} \le \int_{n-1}^{n} \frac{1}{t^x} dt$$

$$\Rightarrow \frac{1}{n^x} \le \int_{n-1}^{n} \frac{1}{t^x} dt$$

$$\sum_{n=2}^{N} \frac{1}{n^x} \le \sum_{n=2}^{N} \int_{n-1}^{n} \frac{1}{t^x} dt = \int_{1}^{N} \frac{1}{t^x} dt$$

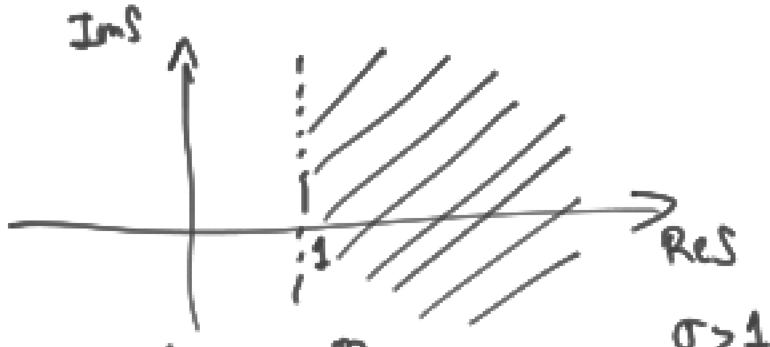
$$= \frac{t^{-x+1}}{-x+1} \Big|_{t=1}^{t=N} = \frac{N^{-x+1}}{-x+1} + \frac{1}{x-1}$$

$-x+1 < 0$ $\quad N \to \infty$ $\quad 0$

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \text{ is convergent for } x > 1$$

$x \leadsto s$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$



$|n^s| = n^{\operatorname{Re} s}$

$s = \sigma + it$

$\sigma > 1$

$\underline{\operatorname{Re} s > 1}$: $\quad \sum_{n=1}^{\infty} \left\| \frac{1}{n^s} \right\| = \underline{\sum_{n=1}^{\infty} \frac{1}{n^\sigma}}$ is finite

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ ; defined in } \operatorname{Re} s > 1$$

ζ is ABSOLUTELY CONVERGENT in $\operatorname{Re} s > 1$.

$$\zeta_N(s) = \sum_{n=1}^{N} \frac{1}{n^s} \text{ ; analytic in } \operatorname{Re} s > 1$$

$\zeta_N(s) \longrightarrow \zeta(s)$; when $N \to \infty$, for any fixed $s \in \mathbb{C}$, $\operatorname{Re} s > 1$.

Let $\varepsilon > 0$; consider $H_\varepsilon = \{ s \in \mathbb{C} : \operatorname{Re} s \ge 1 + \varepsilon \}$
(Fixed)

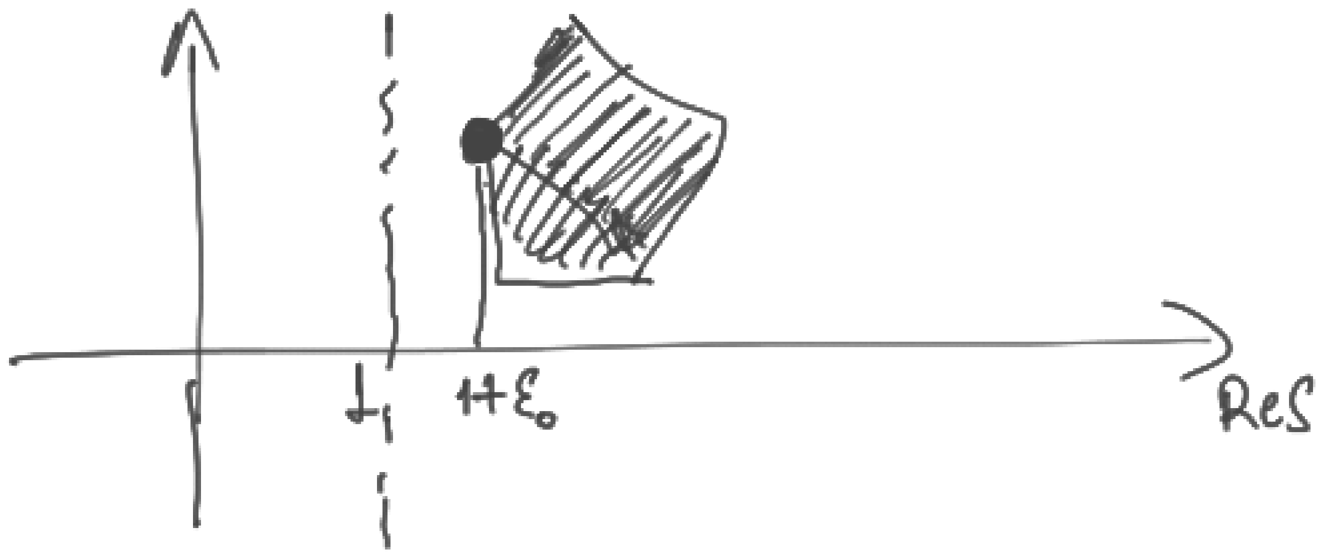
$\zeta_N \longrightarrow \zeta$ uniformly in H $\quad \underline{\underline{\sigma \ge 1 + \varepsilon}}$

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^\sigma} \le \frac{1}{n^{1+\varepsilon}} \qquad \sum_{n \ge 1} \frac{1}{n^{1+\varepsilon}} \text{ is convergent}$$

$\Rightarrow$ T. Weierstrass $\quad \sum_{n=1}^{N} \frac{1}{n^s}$ is uniformly convergent in H

$\zeta_N(s) \xrightarrow[H]{\text{uniformly}} \zeta(s)$ $\quad$ $\zeta_N(s)$ is uniformly con. in H

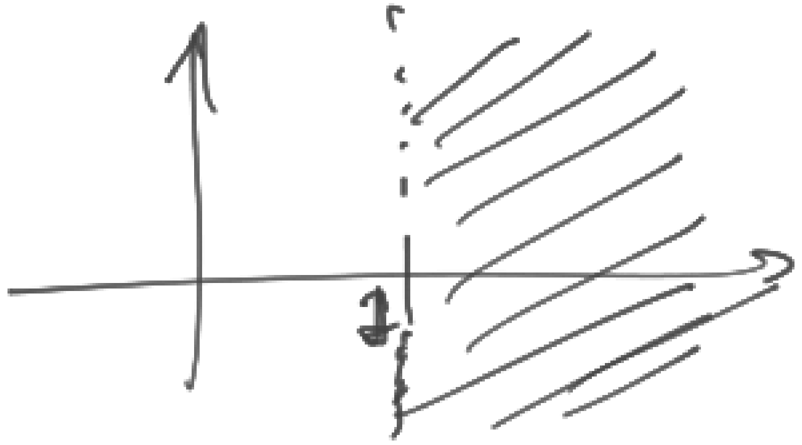
Let K a compact set in $\operatorname{Re} s > 1$



$$K \subset \{ s \in \mathbb{C} : \operatorname{Re} s \geq 1 + \varepsilon_0 \}$$

$\zeta_N(s) \xrightarrow[\text{(in } K)]{\text{uniformly}} \zeta(s)$ in H_{ε_0} (in particular uniformly in K)

ζ is an analytic function in $\operatorname{Re} s > 1$.



$$\lim_{\sigma \to 1^+} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} = \infty$$

$$\tilde{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

PROOF: $\tilde{\zeta}_N(s) \to \tilde{\zeta}(s)$ uniformly in compacts of $\operatorname{Re} s > 0$

$\tilde{\zeta}$ is an analytic function in $\operatorname{Re} s > 0$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^s} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \qquad \sigma > 1$$

$$\left. \sum_{n=1}^{\text{“}\infty\text{”}} \frac{1}{n^{\sigma}} \; ; \; \sigma \leq 1 \right\} \text{ is NO CONVERGENT}$$

$$\sum_{n=1}^{\text{“}\infty\text{”}} \frac{1}{\sqrt{n}} \text{ is NOT CONVERGENT}$$

$\zeta_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}$; $\tilde{\zeta}_N(s) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}$ $\quad \zeta_N(s) \xrightarrow{Re\, s>1} \zeta(s)$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$; $\tilde{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ $\quad \tilde{\zeta}_N(s) \to \tilde{\zeta}(s)$

Re s > 1

$$\zeta_{2N}(s) - \tilde{\zeta}_{2N}(s) = \sum_{n=1}^{2N} \frac{1-(-1)^{n+1}}{n^s} = \left.\sum_{\substack{k=1 \\ \boxed{n=2k-1}}}^{N} \frac{1-(-1)^{2k-1+1}}{(2k-1)^s} \right\} \text{“}0\text{”}$$

$$+ \sum_{\substack{k=1 \\ \boxed{n=2k}}}^{N} \frac{1-(-1)^{2k+1}}{(2k)^s} \quad 1-(-1) \checkmark$$

$$= \sum_{k=1}^{N} \frac{2}{(2k)^s}$$

$$\zeta_{2N}(s) - \tilde{\zeta}_{2N}(s) = \frac{2}{2^s} \sum_{k=1}^{N} \frac{1}{k^s} = 2^{1-s} \cdot \zeta_N(s)$$

$$\zeta_{2N}(s) - \tilde{\zeta}_{2N}(s) = 2^{1-s} \zeta_N(s), \quad \text{for } Re\, s > 1$$

$\downarrow \qquad \downarrow N \to \infty$

$$\zeta(s) - \tilde{\zeta}(s) = 2^{1-s} \cdot \zeta(s)$$

$$\zeta(s) - 2^{1-s} \zeta(s) = \tilde{\zeta}(s)$$

$$\boxed{\zeta(s)(1-2^{1-s}) = \tilde{\zeta}(s)} \; ; Re\, s > 1$$

$$\boxed{\zeta(s) = \frac{\tilde{\zeta}(s)}{1-2^{1-s}}} \quad Re\, s > 1$$

EXTENDS ζ FOR Re s > 0

$$\boxed{1 - 2^{1-s} = 0} \Leftrightarrow 1 = 2^{1-s}$$

$s = 1$

POLE

$\lambda > 0$:

- $G_\lambda(t) = e^{-\pi\lambda t^2} \Rightarrow \widehat{G_\lambda}(x) = \lambda^{-1/2} e^{-\pi \frac{x^2}{\lambda}}$

PSM: $\sum_{n\in\mathbb{Z}} G_\lambda(x+n) = \sum_{k\in\mathbb{Z}} \widehat{G_\lambda}(k) e^{2\pi i k x}$ $\quad x=0$

$$\boxed{\sum_{n\in\mathbb{Z}} G_\lambda(n) = \sum_{k\in\mathbb{Z}} \widehat{G_\lambda}(k)}$$

$$\sum_{n\in\mathbb{Z}} G_\lambda(n) = \sum_{n\in\mathbb{Z}} e^{-\pi\lambda n^2} = 1 + 2\underbrace{\sum_{n\geq 1} e^{-\pi\lambda n^2}} = 1 + 2\Theta(\lambda)$$

$$\sum_{k\in\mathbb{Z}} \widehat{G_\lambda}(k) = \lambda^{-1/2}\Big(1 + 2\underbrace{\sum_{k\geq 1} e^{-\pi\frac{k^2}{\lambda}}}\Big)$$

$$= \lambda^{-1/2}\left(1 + 2.\Theta\left(\tfrac{1}{\lambda}\right)\right)$$

$$\underbrace{1 + 2\Theta(\lambda) = \lambda^{-1/2}\left(1 + 2\Theta\left(\tfrac{1}{\lambda}\right)\right)}; \quad \Theta(\lambda) = \sum_{n\geq 1} e^{-\pi\lambda n^2}$$

$\lambda < 1$: $1 + 2\Theta(\lambda) \leq \lambda^{-1/2}\left(1 + 2.\frac{e^{-\pi/\lambda}}{1-e^{-\pi}}\right) \leq \lambda^{-1/2}\left(1 + \frac{2}{1-e^{-\pi}}\right) = \lambda^{-1/2}\left(\frac{3-e^{-\pi}}{1-e^{-\pi}}\right)$

$\lambda > 1$: $0 < \Theta(\lambda) = \sum_{n\geq 1} e^{-\pi\lambda n^2} \leq \sum_{n\geq 1} e^{-\pi\lambda n} = \frac{e^{-\pi\lambda}}{1-e^{-\pi\lambda}}$

$\lambda > 1 \Rightarrow -\lambda < -1 \Rightarrow -\pi\lambda < -\pi$

$e^{-\pi\lambda} < e^{-\pi}$

$-e^{-\pi\lambda} > -e^{-\pi} \Rightarrow 1-e^{-\pi\lambda} > 1-e^{-\pi}$

$\frac{1}{1-e^{-\pi\lambda}} < \frac{1}{1-e^{-\pi}}$

$\lambda > 1$: $\boxed{0 < \Theta(\lambda) \leq \frac{e^{-\pi\lambda}}{1-e^{-\pi}}}$

$\frac{1}{\lambda}$

$\boxed{\bullet\ \lambda > 1:\ 0 < \Theta(\lambda) \leq C e^{-\pi\lambda},\ \text{for some constant}}$

When $\lambda < 1$; so $\frac{1}{\lambda} > 1$

$\lambda \leq 1$:

$$1 + 2\theta(\lambda) \leq \lambda^{-1/2} \left(\frac{3 - e^{-\pi}}{1 - e^{-\pi}} \right)$$

$2\theta_\lambda \leq$

$$2\theta(\lambda) \leq \lambda^{-1/2} \cdot C$$

$$\theta(\lambda) \leq \lambda^{-1/2} \cdot \frac{C}{2}$$

$$0 < \theta(\lambda) \leq D \cdot \lambda^{-1/2} \quad ; \quad \lambda \leq 1$$

Conclussion:

$\lambda > 1$: $\quad 0 < \theta(\lambda) \leq C \cdot e^{-\pi\lambda}$

$\lambda \leq 1$; $\quad 0 < \theta(\lambda) \leq D \lambda^{-1/2}$

for some $C, D > 0$ $\longrightarrow$

Re $s > 0$:

Fix $n \geq 1$

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

$t = \pi \lambda n^2$

$$\Gamma(s) = \int_0^\infty e^{-\pi\lambda n^2} \cdot (\pi \lambda n^2)^{(s-1)} \cdot \pi n^2 d\lambda$$

$$= \int_0^\infty e^{-\pi\lambda n^2} \cdot \pi^s \cdot \lambda^{s-1} \cdot n^{2(s-1)+2} d\lambda$$

$$\Gamma(s) = \int_0^\infty e^{-\pi\lambda n^2} \cdot \pi^s \cdot \lambda^{s-1} \cdot n^{2s} d\lambda$$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-\pi\lambda n^2} \cdot \pi^{\frac{s}{2}} \lambda^{\frac{s}{2}-1} \cdot n^{s} d\lambda$$

Re $s > 0$: $\pi^{-s/2} \cdot \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty e^{-\pi\lambda n^2} \cdot \lambda^{\frac{s}{2}-1} d\lambda$, $n \geq 1$

In particular for Re $s > 1$: Summing for $1 \leq n$:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n \geq 1} \int_0^\infty e^{-\pi\lambda n^2} \cdot \lambda^{s/2-1} d\lambda$$