Class 35: Riemann zeta-function and Uncertainty principle

Remember:

*) $\Gamma(s)$ is a meromorphic function with simple poles at $s = 0, -1, -2, -3, \dots$

$\Gamma(0+1) = \Gamma(1) = 1$

$\Gamma(s+1) = s\Gamma(s) \Rightarrow \dfrac{\Gamma(s+1)}{s} = \Gamma(s)$

$\operatorname{Re} s > 0$ $\qquad$ $\operatorname{Re} s > -1$

SIMPLE POLE

Γ has zeros?

*) $\dfrac{1}{\Gamma(s)}$ is an entire function with simple zeros at $s = 0, -1, -2, -3, \dots$

$s = 0, -1, -2$

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}} \Rightarrow \frac{\Gamma(s)\sin \pi s}{\pi} = \frac{1}{\Gamma(s)}$$

$0 < \operatorname{Re} s < 1$ $\qquad$ $\sin \pi s$

Define:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \quad \text{analytic in } \operatorname{Re} s > 1.$$

DIRICHLET ETA-FUNCTION $\to$ $$\tilde{\zeta}(s) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^s} \quad \text{analytic in } \operatorname{Re} s > 0$$

$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}$ uniformly on compacts

We have proved:

$$\zeta(s) = \frac{\tilde{\zeta}(s)}{1 - 2^{1-s}}; \quad \operatorname{Re} s > 1$$

$\zeta(s)$ $\qquad$ $\tilde{\zeta}$

1

POLES: $\boxed{1 - 2^{1-s} = 0}$

$1 = 2^{s-1}$

$1 = e^{(s-1)\ln 2}$

$(s-1)\ln 2 = 2\pi k, \ k \in \mathbb{Z}$

$$\boxed{s = \frac{2\pi k}{\ln 2} + 1}\ ? \quad , k \in \mathbb{Z}, \ k \neq 0$$

$k = 0;\ s = 1;\quad \tilde{\zeta}(1) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$

$\dfrac{(-1)^{n+1}}{n^s} + \dfrac{(-1)^{n+2}}{(n+1)^s}$

$\dfrac{1}{n^s} - \dfrac{1}{(n+1)^s}$

$= \dfrac{(n+1)^s - n^s}{n^s (n+1)^s}$

$\approx \dfrac{n^{s-1}}{n^s (n+1)^s}$

$\frac{1}{(s+1)}$ $\quad$ $\operatorname{Re} s > 0$

$$\Theta(\lambda) = \sum_{n\geq 1} e^{-\pi\lambda n^2}$$

C, D $\in \mathbb{R}$

- $\boxed{0 < \Theta(\lambda) \leq C e^{-\pi\lambda}}$, for $\lambda > 1$
- $0 < \Theta(\lambda) \leq D\lambda^{-1/2}$, for $0 < \lambda \leq 1$
- $1 + 2\Theta(\lambda) = \lambda^{-1/2}\left(1 + 2\Theta\left(\frac{1}{\lambda}\right)\right)$ for $\lambda > 0$ (Poisson Summation F.) $\Leftarrow$ MODULAR EQUATION

Res>0: $\boxed{\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt}$ $\downarrow$ $t = \pi\lambda n^2$, $n \geq 1$, "$\lambda > 0$"

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-\pi\lambda n^2} \pi^{\frac{s}{2}} \lambda^{\frac{s}{2}-1} n^s\,d\lambda \quad \Leftarrow \text{ WE HAVE PROVED}$$

For Res>1:

$$\pi^{-s/2}\Gamma(s/2)\, n^{-s} = \int_0^\infty e^{-\pi\lambda n^2} \lambda^{\frac{s}{2}-1}\,d\lambda$$

$\downarrow$ SUMMING OVER ALL $n \geq 1$ ($\sum_{n\geq 1}$, $\sum_{n\geq 1}$)

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n\geq 1}\int_0^\infty e^{-\pi\lambda n^2}\lambda^{\frac{s}{2}-1}\,d\lambda$$

$\downarrow$ CHANGE

$$= \int_0^\infty \sum_{n\geq 1} e^{-\pi\lambda n^2} \lambda^{\frac{s}{2}-1}\,d\lambda$$

$\searrow n$

$$\sum_{n\geq 1}^{N} \int_0^\infty e^{-\pi\lambda n^2}\lambda^{\frac{s}{2}-1}\,d\lambda = \int_0^\infty \sum_{n\geq 1}^{N} e^{-\pi\lambda n^2}\lambda^{\frac{s}{2}-1}\,d\lambda$$

$N \to \infty$ $\qquad$ $N \to \infty$

EXERCISE ✓

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\,d\lambda \quad ; \quad \text{Re}\,s > 1$$

$$= \int_0^1 \Theta(\lambda)\lambda^{\frac{s}{2}-1}\,d\lambda + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\,d\lambda$$

$$= \int_0^1 \frac{1}{2}\left[\lambda^{-\frac{1}{2}}\left(1 + 2\Theta\left(\frac{1}{\lambda}\right)\right) - 1\right]\lambda^{\frac{s}{2}-1}\,d\lambda + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\,d\lambda$$

$\lambda^{-1} = \frac{1}{\lambda} = t$, $\lambda = t^{-1}$

$$= \int_1^\infty \frac{1}{2}\left[t^{1/2}(1 + 2\Theta(t)) - 1\right] \cdot t^{1-\frac{s}{2}}\frac{dt}{t^2} + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\,d\lambda$$

$$= \frac{1}{2}\int_1^\infty \left(t^{1/2} + 2t^{1/2}\Theta(t) - 1\right) t^{-1-\frac{s}{2}}\,dt + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\,d\lambda$$

$$= \frac{1}{2}\left[\int_1^\infty t^{-\frac{1}{2}-\frac{s}{2}}\,dt + \int_1^\infty 2\Theta(t)t^{-\frac{1}{2}-\frac{s}{2}}\,dt - \int_1^\infty t^{-1-\frac{s}{2}}\,dt\right] + \ldots$$

$$= \frac{1}{2}\left[\frac{t^{\frac{1}{2}-\frac{s}{2}}}{\frac{1}{2}-\frac{s}{2}}\Bigg|_1^\infty - \frac{t^{-s/2}}{-s/2}\Bigg|_1^\infty\right] + \int_1^\infty \Theta(\lambda)\lambda^{-\frac{1}{2}-\frac{s}{2}}\,d\lambda + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\,d\lambda$$

$$= \frac{1}{2}\left[-\frac{1}{\frac{1}{2}-\frac{s}{2}} - \left\{-\frac{1}{-s/2}\right\}\right] = \frac{1}{s-1} - \frac{1}{s} + \ldots + \ldots$$

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \underbrace{\int_1^{\infty} \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda}_{g(s)}$$

$\mathrm{Re}\, s > 1$

$$g(s) = \int_1^{\infty} \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda$$

$$g_n(s) = \int_1^{n} \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda$$

- g_n is continuous $\quad g_n(s) - g_n(s_0) \to 0$
- $$\int_{\partial\Delta} g_n(s)\,ds = \int_{\partial\Delta}\left(\int_1^n \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda\right)ds$$
$$= \int_1^n \theta(\lambda) \underbrace{\int_{\partial\Delta}\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)ds}_{0}\,d\lambda = 0$$

$\Longrightarrow$ MORERA TH. $\Rightarrow$ g_n is analytic in $\mathbb{C}$

g_n is entire

g is an entire function (analytic on $\mathbb{C}$)

$$\zeta(s) = \left(\frac{1}{s-1} - \frac{1}{s} + g(s)\right)\pi^{s/2} \cdot \frac{1}{\Gamma\left(\frac{s}{2}\right)}, \quad \mathrm{Re}\, s > 1$$

$\mathbb{C} \quad \mathbb{C} \quad \mathbb{C} \quad \mathbb{C} \quad \mathbb{C}$

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \quad \mathrm{Re}\, s > 1$$

$\frac{1}{s} \cdot \frac{1}{\Gamma\left(\frac{s}{2}\right)}$ has a zero at $s=0$

$\frac{1}{s\Gamma(s/2)}$

$\cancel{\frac{1/\Gamma(s/2)}{s}}$ doesn't have zero at $s=0$

- ζ is meromorphic function in $\mathbb{C}$.
- ζ has a simple pole at $s=1$.
- ζ has zeros at $s=-2,-4,-6,\ldots$ (zeros of $1/\Gamma(s/2)$).

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \boxed{\frac{1}{s-1} - \frac{1}{s} + g(s)}$$

$\rightarrow \frac{1}{(1-s)-1} - \frac{1}{1-s} + g(1-s)$

$-\frac{1}{s} + \frac{1}{s-1} + g(s)$ in $\operatorname{Re} s > 1$

where $g(s) = \int_1^{\infty} \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda$ — entire function

- ζ has a meromorphic continuation in $\mathbb{C}$.
- ζ has only a simple pole at $s=1$.

- $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \cdots \frac{1}{n^s} + \cdots = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \cdots\right)$

$\operatorname{Re} s > 1$

$\left|\frac{1}{p^s}\right| = \frac{1}{p^{\operatorname{Re} s}} \leq \frac{1}{p}$

$= \prod_p \frac{1}{1-\frac{1}{p^s}} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$

$\underbrace{1 + x + x^2 + \cdots}_{x}$

uniform in compacts

$\zeta(s)$ has no zeros in $\operatorname{Re} s > 1$

$\left|\frac{1}{p^s}\right| < 1$

$$g(1-s) = \int_1^{\infty} \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{(1-s)}{2}} + \lambda^{\frac{1-s}{2}-1}\right)d\lambda = \underbrace{\int_1^{\infty} \theta(\lambda)\left(\lambda^{\frac{-1+s}{2}} + \lambda^{\frac{-1-s}{2}}\right)d\lambda}_{= g(s)}$$

$g(1-s) = g(s)$ for $s \in \mathbb{C}$

$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + g(s)$ holds for $s \in \mathbb{C}$

$$\boxed{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}$$ for all $s \in \mathbb{C}$ → FUNCTIONAL EQUATION

$A = B$

- $\zeta(s)$ has no zeros in $\operatorname{Re} s > 1$.

- In $\operatorname{Re} s < 0$:

$\operatorname{Re}(1-s) > 1$

$\operatorname{Re}\left(\frac{1-s}{2}\right) > \frac{1}{2}$

$\operatorname{Re} s < 0$

$\Gamma\left(\frac{1-s}{2}\right)$ is analytic in $\operatorname{Re} s < 0$;

$\zeta(1-s)$. is analytic in $\operatorname{Re} s < 0$

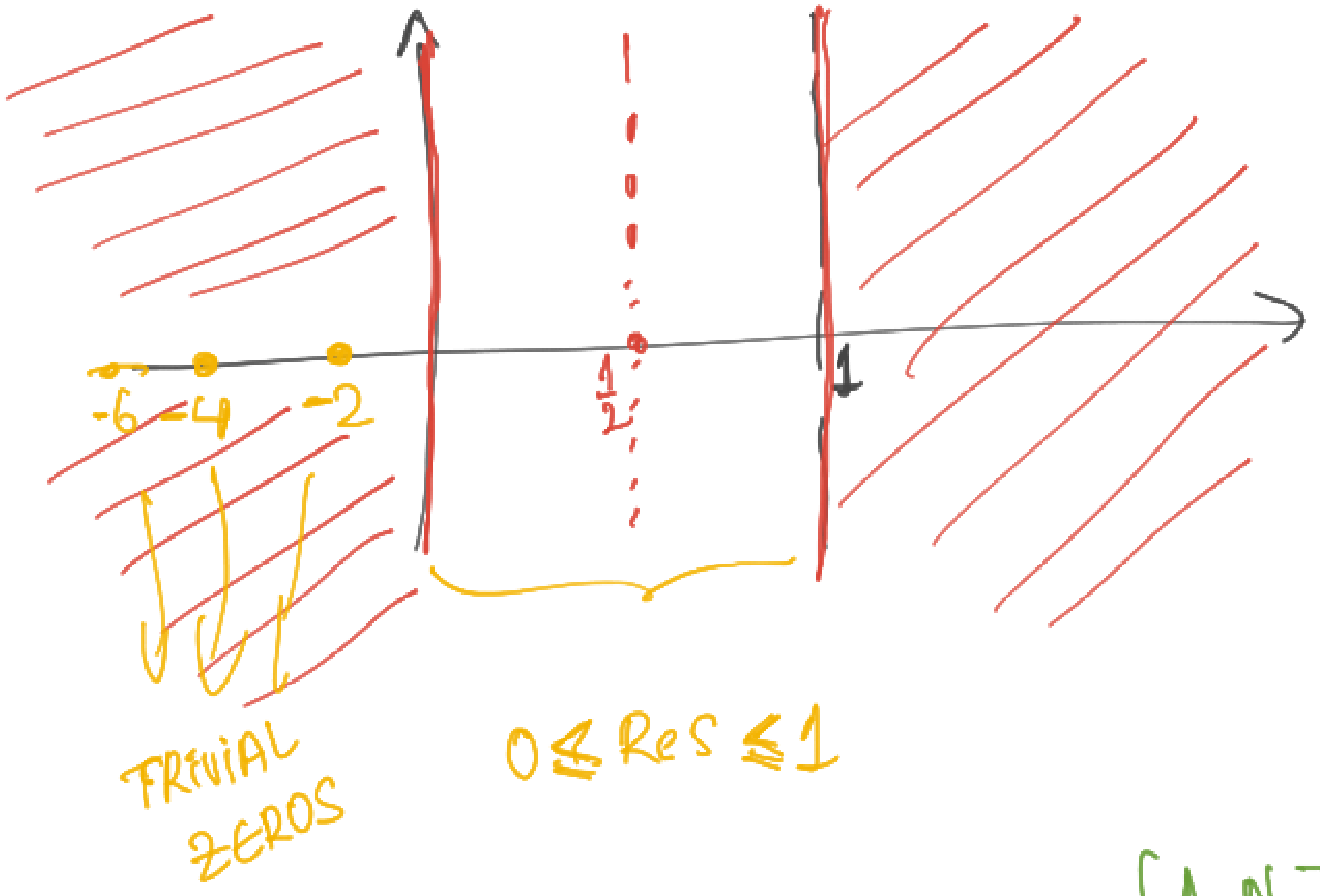
⇒ B is analytic in $\operatorname{Re} s < 0$

⇒ A is analytic in $\operatorname{Re} s < 0$

$\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is analytic in $\operatorname{Re} s < 0$ → ζ has zeros in $s = -2, -4, -6, \ldots$

SIMPLE ZEROS

poles in $s = -2, -4, -6, \ldots$



- $\operatorname{Re} s = 1$, ζ has no zeros (A.N.T.)
- $\operatorname{Re} s = 0$, ζ has no zeros (functional equation)

$\boxed{16:15\ \text{PM} - 17:00}$

$0 < \operatorname{Re} s < 1$ → CRITICAL STRIP

1859

RIEMANN HYPOTHESIS

ALL THE ZEROS (NON-TRIVIAL ZEROS) IN $0 < \operatorname{Re} s < 1$ HAS $\operatorname{Re} s = \frac{1}{2}$.

10^{13}

$\operatorname{Im}(s) < 10^{13}$ → 10^{14} → 10^{15}

$\zeta(s) = 0$; $\operatorname{Im}(s) < 10^{13} \Rightarrow \operatorname{Re} s = \frac{1}{2}$

\$ 1000 000 CLAY