Class 37: Uncertainty principle

$f \in M(\mathbb{R}) \longrightarrow \hat{f}$ is a cont. function

$\hat{f}$ is bounded

$\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0$

Theorem: Let $f \in S(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$. Then:

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right) \cdot \left(\int_{-\infty}^{\infty} t^2 |\hat{f}(t)|^2 dt\right) \geq \frac{1}{16\pi^2}.$$

The equality holds if and only if $f(x) = Ae^{-Bx^2}$, with $B>0$; $|A|^2 = \sqrt{\frac{2B}{\pi}}$.

$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$

Suppose f has compact support in $[-\delta, \delta]$

$$\int_{-\infty}^{\infty} x^2 |f(x)|^2 = \int_{-\delta}^{\delta} x^2 |f(x)|^2 dx \leq \delta^2 \int_{-\infty}^{\infty} |f(x)|^2 dx = \delta^2 \xrightarrow{\delta \to 0} 0$$

$|x| < \delta$

Suppose $\hat{f}$ has compact support in $[-\varepsilon, \varepsilon]$

$$\int_{-\infty}^{\infty} t^2 |\hat{f}(t)|^2 dt \leq \varepsilon^2 \xrightarrow{\varepsilon \to 0}$$

$\varepsilon \to 0$

$\frac{1}{1+x^2}$

$e^{-2\pi|x|}$

PROOF:

$$1 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 \cdot (x)' dx$$

$$\overset{\text{I.P.}}{=} \underbrace{|f(x)|^2 \cdot x \Big|_{x=-\infty}^{x=\infty}}_{0} - \int_{-\infty}^{\infty} \frac{d}{dx}(|f(x)|^2) \cdot x \, dx$$

$$= -\int_{-\infty}^{\infty} \frac{d}{dx}(|f(x)|^2) \cdot x \, dx$$

$$\frac{d}{dx}(|f(x)|^2) = \frac{d}{dx}(f(x) \cdot \overline{f(x)}) = f'(x)\overline{f(x)} + f(x)\overline{f'(x)}$$

$$1 = -\int_{-\infty}^{\infty} (f'(x)\overline{f(x)} + f(x)\overline{f'(x)}) \cdot x \, dx$$

$\left|\int_{-\infty}^{\infty} g(x) dx\right| \leq \int_{-\infty}^{\infty} |g(x)| dx$

$$1 = \left|\int_{-\infty}^{\infty} f'(x)\overline{f(x)} \cdot x + f(x)\overline{f'(x)} \cdot x \, dx\right|$$

$$\leq \int_{-\infty}^{\infty} |f'(x)\overline{f(x)}\, x + f(x)\overline{f'(x)}\, x| \, dx$$

$$\leq 2\int_{-\infty}^{\infty} |f'(x)\overline{f(x)}\, x| \, dx$$

$\left(\int_{-\infty}^{\infty} |fg| \leq \left(\int_{-\infty}^{\infty} |f|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |g|^2 dx\right)^{1/2}\right)$

C.S.

$$\leq 2\left(\int_{-\infty}^{\infty} |f(x)\, x|^2 dx\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} |f'(x)|^2 dx\right)^{1/2}$$

Plancherel

$$= 2\left(\int_{-\infty}^{\infty} |f(x)\, x|^2 dx\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} |\widehat{f'}(x)|^2 dx\right)^{1/2}$$

F.T of the derivative

$$= 2\left[\int_{-\infty}^{\infty} |f(x)\, x|^2 dx\right]^{1/2} \cdot \left(\int_{-\infty}^{\infty} |2\pi i x \hat{f}(x)|^2 dx\right)^{1/2}$$

$$= 2\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} |2\pi i t \hat{f}(t)|^2 dt\right)^{1/2}$$

$$= 2\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)^{1/2} \cdot 2\pi \left(\int_{-\infty}^{\infty} t^2 |\hat{f}(t)|^2 dt\right)^{1/2}$$

$$1 \leq 4\pi \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} t^2 |\hat{f}(t)|^2 dt\right)^{1/2}$$

$$\frac{1}{16\pi^2} \leq \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} t^2 |\hat{f}(t)|^2 dt\right)$$

$f'(x) = \lambda \cdot f(x) \cdot x$, for some $\lambda \in \mathbb{C}$

f is real valued $\Rightarrow \lambda \in \mathbb{R}$

$$\boxed{f'(x) = \lambda f(x) \cdot \cancel{x}} \quad \forall x \in \mathbb{R}$$

$\leq \leq \to$ yellow

green

$$\left(f(x) e^{-\lambda \frac{x^2}{2}}\right)' = f'(x) e^{-\lambda\frac{x^2}{2}} + f(x) \cdot e^{-\lambda \frac{x^2}{2}} (-\lambda x)$$

$$= e^{-\frac{\lambda x^2}{2}} \left(f'(x) - \lambda x f(x)\right)$$

0

$= 0$

$$\boxed{f(x) = A e^{\lambda \frac{x^2}{2}}} \quad . \quad A \in \mathbb{R}$$

$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1 \Rightarrow \lambda < 0; \quad \lambda = -2B, B > 0$

$$\boxed{f(x) = A e^{-Bx^2}}$$

$$\int_{-\infty}^{\infty} |A e^{-Bx^2}|^2 dx = 1 \Rightarrow |A|^2 \int_{-\infty}^{\infty} e^{-2Bx^2} d\cancel{x} = 1$$

$\sqrt{2B}\, x = \sqrt{\pi}\, u$

$$\Rightarrow |A|^2 \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\pi u^2}}_{1} \frac{\sqrt{\pi}\, du}{\sqrt{2B}} = 1$$

$$\Rightarrow \boxed{|A|^2 = \sqrt{\frac{2B}{\pi}}}$$

$$\boxed{f(x) = A e^{-Bx^2}} , \quad \boxed{B > 0, \quad |A|^2 = \sqrt{\frac{2B}{\pi}}}$$

Corollary: Let $f \in S(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$.

Then, for any $x_0, t_0 \in \mathbb{R}$

$$\left(\int_{-\infty}^{\infty} (x-x_0)^2 |f(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} (t-t_0)^2 |\hat{f}(t)|^2 dt\right) \geq \frac{1}{16\pi^2}$$

PROOF:

Use $\boxed{g(x) = e^{-2\pi i x t_0} \cdot f(x+x_0) \text{ in Theorem}}$

PALEY - WIENER THEOREM

$f: \mathbb{R} \to \mathbb{C}$

Theorem: Let $f \in M(\mathbb{R})$, and $M > 0$.

The following propositions are equivalent:

i) f has an extension as an entire function such that $|f(z)| \leq A e^{2\pi M |z|}$ some $A > 0$ $\forall z \in \mathbb{C}$.

ii) $\hat{f}(t) = 0$ for $|t| \geq M$ (i.e. $\operatorname{supp} \hat{f} \subset [-M, M]$)

PROOF: ii) $\to$ i)

$f \in M(\mathbb{R})$; $\hat{f}(t) = 0$, $\forall |t| \geq M$

$\Rightarrow \hat{f} \in M(\mathbb{R})$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i x t} dt$$

$$f(x) = \int_{-M}^{M} \hat{f}(t) e^{2\pi i x t} dt$$

Let $g(z) = \int_{-M}^{M} \hat{f}(t) e^{2\pi i z t} dt$

$|g(z+h) - g(z)| = \left| \int_{-M}^{M} \hat{f}(t) e^{2\pi i z t} (e^{2\pi i h t} - 1) dt \right|$

$|h| \leq 1$

$h \in \mathbb{C}$

$\leq \int_{-M}^{M} |\hat{f}(t)| |e^{2\pi i z t}| |e^{2\pi i h t} - 1| dt$

$\lim_{s \to 0} \frac{e^{2\pi i s} - 1}{s} = 2\pi i \Rightarrow |e^{2\pi i s} - 1| \leq C_1 |s|$

$\to |s| \leq \delta$

$\frac{e^{2\pi i s} - 1}{s}$

$\delta \leq |s| \leq M$

$|e^{2\pi i s} - 1| \leq C_2 |s|$

$(|t| \leq M)$

$|e^{2\pi i s} - 1| \leq C_3 |s|$

$\leq \int_{-M}^{M} |\hat{f}(t)| e^{|2\pi i z t|} \cdot C_3 |ht| dt \quad \forall \quad |s| \leq M$

$\leq C_3 \int_{-M}^{M} |\hat{f}(t)| e^{2\pi |z| \cdot M} \cdot |h| |t| dt = |h|$

g is continuous

- g is continuous
- $\int_{\partial\Delta} g(z)\,dz = 0$

$$\int_{\partial\Delta}\left(\int_{-M}^{M}\hat{f}(t)\, e^{2\pi i z t}\,dt\right)dz = \int_{-M}^{M}\Big(\hat{f}(t)\underbrace{\int_{\partial\Delta} e^{2\pi i z t}\,dz}_{0}\Big)dt = 0$$

$\Rightarrow$ g is an entire function.

$$g(z) = \int_{-M}^{M}\hat{f}(t)\, e^{2\pi i z t}\,dt$$

$$|g(z)| \le \int_{-M}^{M} |\hat{f}(t)|\, e^{2\pi |z| . M}\,dt$$

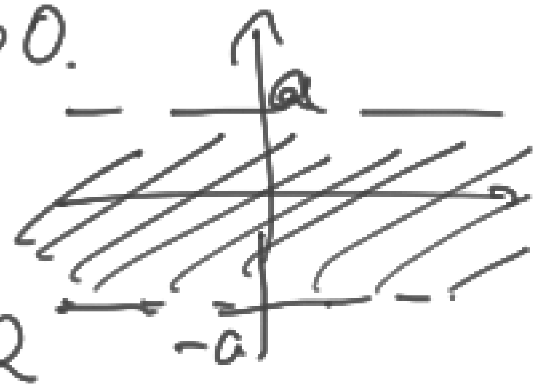
$$= \underbrace{\left(\int_{-M}^{M}|\hat{f}(t)|\,dt\right)}_{A} . \; e^{2\pi M|z|}$$

-) g is an entire function
-) $|g(z)| \le A .\, e^{2\pi M|z|}$
-) $g(x) = \int_{-M}^{M}\hat{f}(t)\, e^{2\pi i x t}\,dt = f(x)$ $\quad \forall x\in\mathbb{R}$

i)

$\boxed{i) \rightarrow ii)}$

Lemma: Let g be an holomorphic function in the strip $\{s\in\mathbb{C} : |\mathrm{Im}(s)| \le a\}$, for some $a>0$.



Suppose that:

$$|g(x+iy)| \le \frac{C}{1+|x|^{1+\delta}} \quad \text{for } x\in\mathbb{R},\ |y|<a$$

then: $|\hat{g}(\xi)| \le B\, e^{-2\pi b|\xi|}$, for some $B>0$, $\xi\in\mathbb{R}$, and any $0\le b<a$.

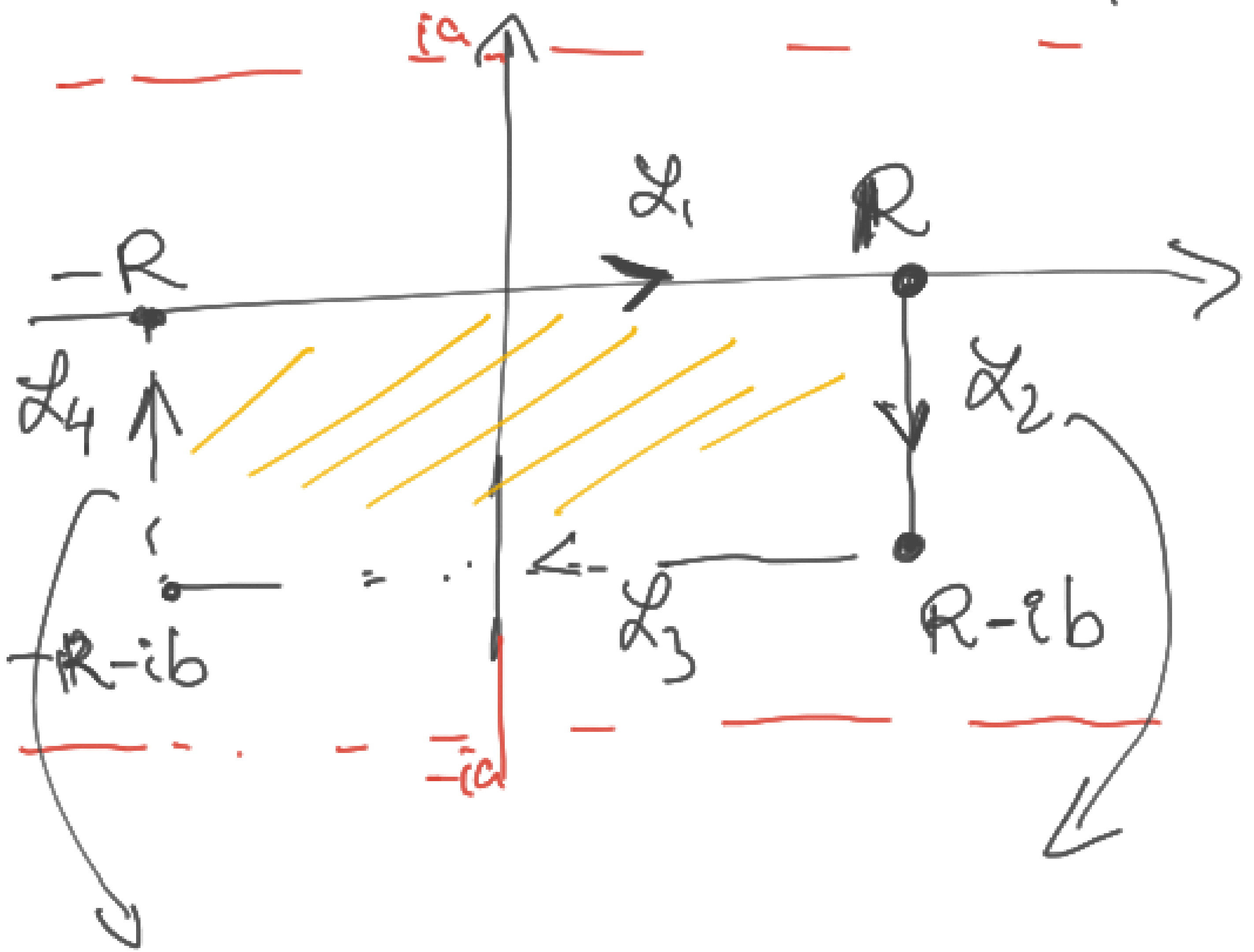
Lemma PROOF

$0<b<a$ (when $b=0$, $\hat{g}$ is bounded)

Assume $\xi>0$.

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)\, e^{-2\pi i x\xi}\, dx$$

$$f(z) = g(z)\, e^{-2\pi i z\xi}$$, f is holomorphic in $|\mathrm{Im}\, z| < a$



$$\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f = 0$$

(f is holomorphic in $\square$)

$$\int_{\gamma_4} f(z)\,dz = \int_{-R-ib}^{-R} f(z)\,dz \qquad z=-R+ix$$

$$= \int_{-b}^{0} f(-R+ix)\cdot i\,dx$$

$$\int_{\gamma_4} f(z)\,dz = \int_{-b}^{0} g(-R+ix)\, e^{-2\pi i (R+ix)\xi}\cdot i\,dx$$

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \int_{-b}^{0} |g(-R+ix)|\, e^{2\pi x\xi}\,dx$$

$$\le \int_{-b}^{0} \frac{c}{1+R^{1+\delta}}\, e^{2\pi x\xi}\,dx \le \frac{C\cdot \int_{-b}^{0} e^{2\pi x\xi}}{R^{1+\delta}}$$

$R\to\infty$ $\qquad \int_{\gamma_4} f(z)\,dz \to 0$ (also for γ_2)

$$\int_{\mathcal{L}_1} f(z)\,dz = -\int_{\mathcal{L}_3} f(z)\,dz \qquad R\to\infty$$

$$\int_{-\infty}^{\infty} g(x)\, e^{-2\pi i x \xi}\, dx = \lim_{R\to\infty} -\int_{R-ib}^{-R-ib} g(z)\, e^{-2\pi i z \xi}\, dz$$

$$\hat{g}(\xi) = \lim_{R\to\infty} \int_{-R-ib}^{R-ib} g(z)\, e^{-2\pi i z \xi}\, dz$$