Class 38: Paly-Wiener Theorem

Theorem: Let $f \in \mathcal{M}(\mathbb{R})$. Are equivalent:

i) f has an extension as entire function such that

$$|f(z)| \le A e^{2\pi M |z|}, \quad A>0, \ M>0, \ \forall z \in \mathbb{C}$$

ii) $\hat{f}(t) = 0 \quad \forall |t| \geq M$. (supp $\hat{f} \subset [-M, M]$)

PROOF:

ii) $\to$ i) ✓

i $\to$ ii)

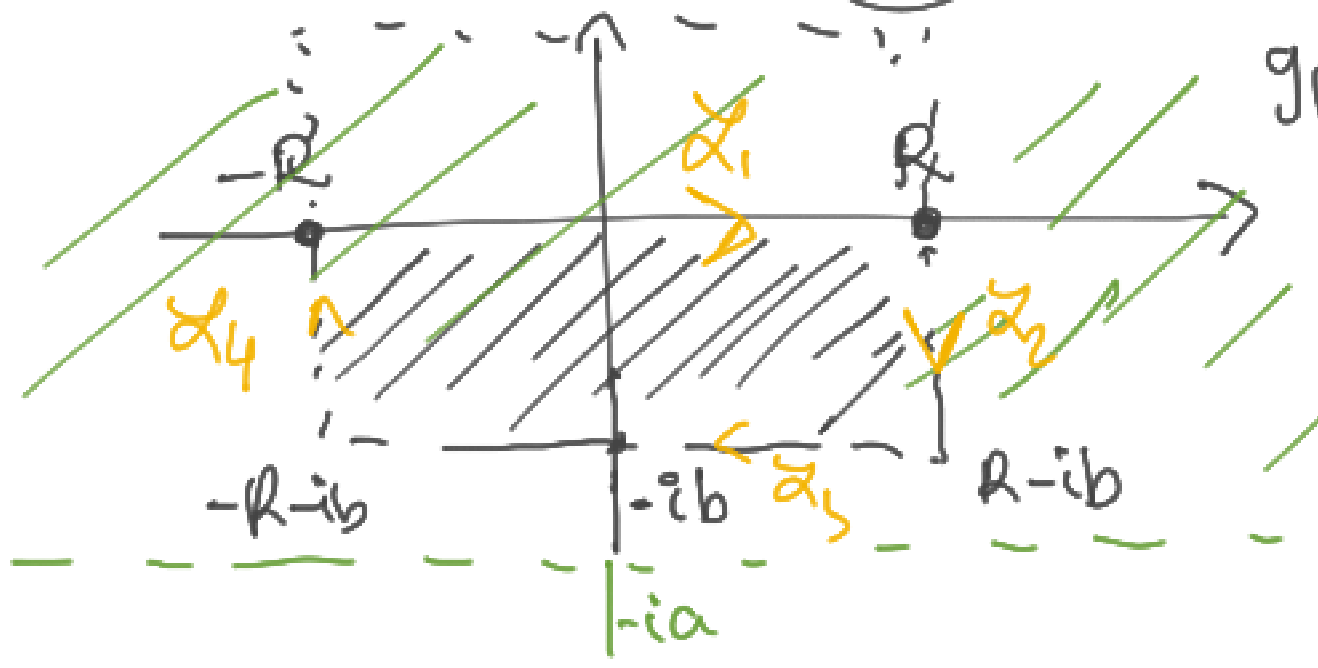
Lemma 1: Let f be a holomorphic function in $\{s \in \mathbb{C} : |\operatorname{Im} s| \le a\}$ such that

$$|f(x+iy)| \le \frac{A}{1+x^2}, \quad \forall x \in \mathbb{R}, \ |y|<a \quad \ldots (*)$$

Then, for any $0 \le b < a$ we have

$$|\hat{f}(t)| \le B e^{-2\pi b|t|}, \quad \forall t \in \mathbb{R}$$

PROOF Lemma 1: Let $t>0$, and $b>0$



$g(z) = f(z) e^{-2\pi i t z}$

$$\int_{\gamma_1} g + \int_{\gamma_2} g + \int_{\gamma_3} g + \int_{\gamma_4} g = 0$$

$$\int_{-R}^{R} f(x) e^{-2\pi i t x} + \int_{\gamma_2} + \int_{\gamma_4} = \int_{-R-ib}^{R-ib} f(z) e^{-2\pi i t z} \, dz$$

$z = x - ib$

$R \to \infty$

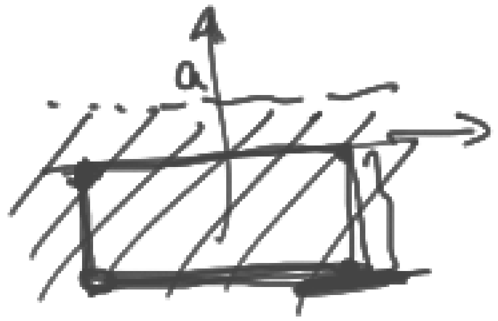
$$\boxed{\hat{f}(t) = \int_{-\infty}^{\infty} f(x-ib) e^{-2\pi i t (x-ib)} \, dx} \leftarrow (*)$$

$$|\hat{f}(t)| \le \int_{-\infty}^{\infty} \frac{A}{1+x^2} e^{-2\pi t b} \, dx = e^{-2\pi t b} \cdot B.$$

$$\Rightarrow |\hat{f}(t)| \le B e^{-2\pi b |t|}$$

Lemma 2: Let f be a holomorphic function in $\{s \in \mathbb{C} : \operatorname{Im} s < a\}$, $a>0$

Suppose that for some $b>0$, $A>0$



$$|f(x+iy)| \leq \frac{A}{1+x^2}; \quad \text{for} \quad -b \leq y < 0$$

Then:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x-ib)\, e^{-2\pi i t(x-ib)}\, dx$$

PROOF. Follows from Lemma 1.

Lemma 3: Let f be a holomorphic function in $\{s \in \mathbb{C} : \operatorname{Im} s < a\}$, for $a>0$. Suppose that

$$|f(x+iy)| \leq \frac{A e^{2\pi M|y|}}{1+x^2}, \quad \text{for } y<0$$

Then: $\hat{f}(t) = 0$, when $t > M$. $(t < -M)$

•) $\hat{f}(t) = 0$ for $t \geq M$

PROOF OF Lemma 3: Let $b>0$ fixed.

Take $-b \leq y < 0$

$$|f(x+iy)| \leq \frac{A e^{2\pi M|y|}}{1+x^2} \leq \frac{A e^{2\pi M b}}{1+x^2} \to A'$$

APPLY Lemma 2:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x-ib)\, e^{-2\pi i t(x-ib)}\, dx$$

$$|\hat{f}(t)| \leq \int_{-\infty}^{\infty} \frac{A e^{2\pi M|b|}}{1+x^2} \cdot e^{-2\pi t b}\, dx$$

$$|\hat{f}(t)| \leq B\, e^{2\pi M b - 2\pi t b} \to B \text{ independent of "b"}$$

$$A\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$|\hat{f}(t)| \leq B\, e^{-2\pi b(t-M)}$$

Fix $t>M$: $\boxed{t-M>0}$ $\quad |\hat{f}(t)| \leq B\, e^{-2\pi b \cdot (t-M)}$

Letting $b \to \infty$ $\quad \hat{f}(t) = 0$

Lemma 4: Let f be an entire function such that $f \in M(\mathbb{R})$ and $|f(x+iy)| \le A e^{2\pi M|y|}$. Then $\hat{f}(t) = 0$ for $t > M$.

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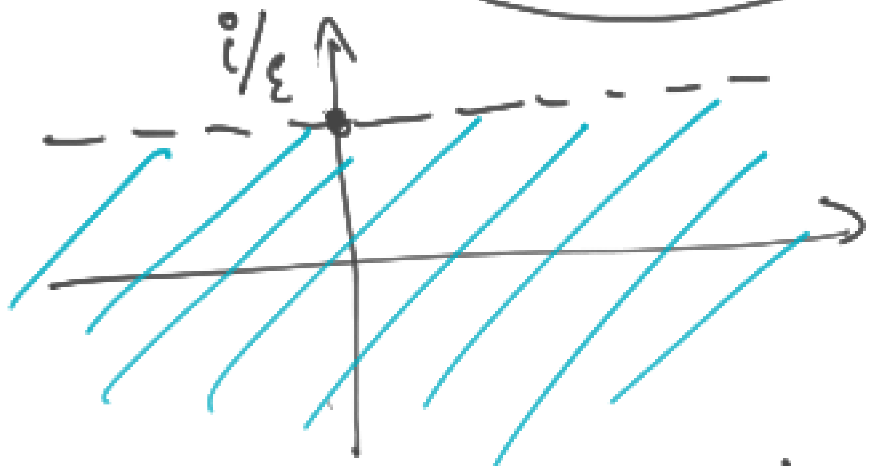
PROOF Lemma 4:

Let $0 < \varepsilon < 1$: (FIXED) ←

Let $f_\varepsilon(z) = \dfrac{f(z)}{(1+i\varepsilon z)^2}$

$1 + i\varepsilon z = 0$

$z = -\dfrac{1}{i\varepsilon} = \dfrac{i}{\varepsilon}$



$$|f_\varepsilon(x+iy)| \le \frac{|f(x+iy)|}{|1+i\varepsilon(x+iy)|^2} \le \frac{A e^{2\pi M|y|}}{|1-\varepsilon y + i\varepsilon x|^2}$$

$$= \frac{A e^{2\pi M|y|}}{(1-\varepsilon y)^2 + \varepsilon^2 x^2} = \frac{(A/\varepsilon^2)\, e^{2\pi M|y|}}{\left(\frac{1}{\varepsilon} - y\right)^2 + x^2}$$

For $y < 0$:

$$\left(\frac{1}{\varepsilon} - y\right)^2 > \frac{1}{\varepsilon^2} > 1$$

$$|f_\varepsilon(x+iy)| \le \frac{A}{\varepsilon^2} \cdot e^{2\pi M|y|} \cdot \frac{1}{1+x^2}$$

$\hat{f}_\varepsilon(t) = 0$, for $t > M$, and any $\varepsilon > 0$. (by Lemma 3).

To conclude we need to prove that $\boxed{\hat{f}_\varepsilon(t) \xrightarrow[\varepsilon \to 0]{} \hat{f}(t).}$

$$|\hat{f}_\varepsilon(t) - \hat{f}(t)| \le \int_{-\infty}^{\infty} |f(x)| \left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| dx$$

$$= \int_{|x| \ge \frac{1}{\varepsilon^{1/8}}} |f(x)| \left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| dx + \int_{|x| \le \frac{1}{\varepsilon^{1/8}}} |f(x)| \left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| dx$$

$$\left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| \le \frac{1}{|1+i\varepsilon x|^2} + 1 = \frac{1}{1+\varepsilon^2 x^2} + 1 \le 2$$

$$\left| \frac{1}{(1+i\varepsilon x)^2} - 1 \right| = \left| \frac{1-(1+i\varepsilon x)^2}{(1+i\varepsilon x)^2} \right| = \left| \frac{-2i\varepsilon x + \varepsilon^2 x^2}{(1+i\varepsilon x)^2} \right| = \frac{\varepsilon\,|-2ix + \varepsilon x^2|}{|1+i\varepsilon x|^2}$$

$$= \varepsilon \frac{|-2ix+\varepsilon x^2|}{|1+i\varepsilon x|^2} = \frac{\varepsilon\cdot\sqrt{4x^2+\varepsilon^2 x^4}}{1+\varepsilon^2 x^2} \le \varepsilon\sqrt{4x^2+\varepsilon^2 x^4}$$

$$|\hat f_\varepsilon(t)-\hat f(t)| \le \int_{|x|\ge 1/\varepsilon^{1/8}} |f(x)|\left|\frac{1}{(1+i\varepsilon x)^2}-1\right|dx + \int_{|x|<1/\varepsilon^{1/8}} |f(x)|\left|\frac{1}{(1+i\varepsilon x)^2}-1\right|dx$$

$$\le 2\int_{|x|\ge 1/\varepsilon^{1/8}} |f(x)|\,dx + \varepsilon\int_{|x|\le 1/\varepsilon^{1/8}} |f(x)|\cdot\sqrt{\frac{4}{\varepsilon^{1/4}}+\frac{\varepsilon^2}{\varepsilon^{1/2}}}\,dx$$

$$= 2\int_{|x|\ge 1/\varepsilon^{1/8}} |f(x)|\,dx + \varepsilon\cdot\sqrt{\frac{4}{\varepsilon^{1/4}}+\varepsilon^{1/2}}\int_{|x|\le 1/\varepsilon^{1/8}} |f(x)|\,dx$$

$$\le 2\int_{|x|\ge 1/\varepsilon^{1/8}} |f(x)|\,dx + \varepsilon\sqrt{\frac{8}{\varepsilon^{1/4}}}\int_{-\infty}^{\infty} |f(x)|\,dx$$

$$= \underbrace{2\int_{|x|\ge 1/\varepsilon^{1/8}} |f(x)|\,dx}_{\to 0} + \sqrt{8}\,\varepsilon^{7/8}\int_{-\infty}^{\infty} |f(x)|\,dx \to 0$$

$\varepsilon\to 0$

$f\in M(\mathbb{R}) \Rightarrow \lim_{M\to\infty}\int_{|x|\ge M} |f(x)|\,dx = 0$

$\hat f_\varepsilon(t) \xrightarrow[\varepsilon\to 0]{} \hat f(t)$, for any $t\in\mathbb{R}$

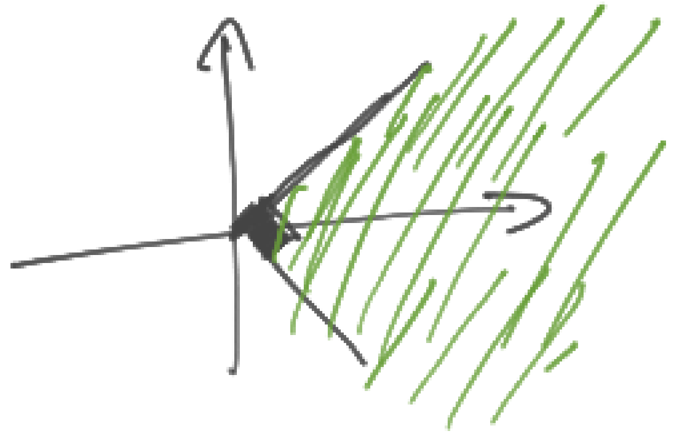
We have proved that, when $t>M \Rightarrow \hat f_\varepsilon(t)=0$

$\downarrow \varepsilon\to 0$

$\hat f(t)=0$

Lemma 5: Phragmén–Lindelöf

Let G be a holomorphic function in $S=\{z\in\mathbb{C}: -\frac{\pi}{4}<\arg z<\frac{\pi}{4}\}$



and G is continuous in $\overline{S}$

If $|G(z)|\le N$ in ∂S and $|G(z)|\le Ce^{B|z|}$ in S.

$\Rightarrow |G(z)|\le N$ in S.

$G(z) := G(ze^{i\pi/4})$

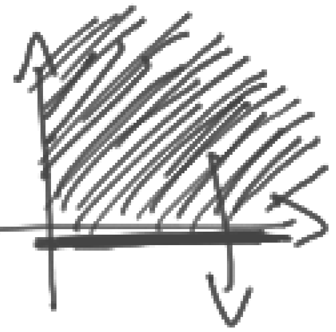
PROOF OF THEOREM: $f \in M(\mathbb{R})$;

f is an entire function such that $|f(z)| \le A e^{2\pi M|z|}$ for $z \in \mathbb{C}$

Define:

$$G(z) = f(z)\, e^{2\pi i M z} \quad (\operatorname{Re} z > 0,\ \operatorname{Im} z > 0)$$

In the 1° quadrant



C_1

$x \in \mathbb{R}$, $x > 0$: $|G(x)| = |f(x)| \le B.$ $\quad (f \in M(\mathbb{R}))$

$x > 0$: $|G(ix)| = |f(ix)|\,|e^{2\pi i M(ix)}|$

$$\le A\, e^{2\pi M|ix|} \cdot |e^{-2\pi M x}|$$

$$= A\, e^{2\pi M x} \cdot e^{-2\pi M x} = A$$

$|G(x)| \le B + A$, $x \in \mathbb{R}$, $x > 0$; $|G(ix)| \le A + B$, $x > 0$

$$\boxed{|G(z)| \le A + B \quad \forall z \in \partial C_1}$$

$\Rightarrow$ Lemma 5:

$$|G(z)| \le A + B \quad \forall z \in C_1$$

$$|f(z)\, e^{2\pi i M z}| \le A + B$$

$$|f(z)\, e^{2\pi i M(x+iy)}| \le A + B$$

$$|f(z)\, e^{-2\pi M y}| \le A + B$$

$$|f(x+iy)| \le C \cdot e^{2\pi M y} \quad \text{in } C_1, \ x > 0, \ y > 0$$

The same proof in the other C_2, C_3, C_4:

$$|f(z)| \le C \cdot e^{2\pi M|y|} \quad z \in \mathbb{C},$$

$f \in M(\mathbb{R})$ $\Rightarrow$ $\hat{f}(t) = 0$ $\forall t \ge M$

Take $g(z) = f(-z)$ $\longrightarrow$ $\hat{f}(-t) = 0$, $t \ge M$

$g \in M(\mathbb{R})$; $|g(z)| \le C e^{2\pi M|z|}$ $\longrightarrow$ $\hat{g}(t) = 0$, $t \ge M$

Conclusion:

$\hat{f}(t) = 0 \quad \forall |t| \geq M$

$|f(z)| \leq A e^{2\pi M|z|} \quad \forall z \in \mathbb{C}$

$\Downarrow$

f has exponential type $\underline{2\pi M}$.

f is an entire function, $f \in \mathcal{M}(\mathbb{R})$

Are equivalent:

- f has exponential type $2\pi M$.
- $\operatorname{supp} \hat{f} \subset [-M, M]$.

If $f \in \mathcal{M}(\mathbb{R})$; f has exponential type π

$\xRightarrow{\text{P.W.}} \operatorname{supp} \hat{f} \subset [-\frac{1}{2}; \frac{1}{2}]$

$\Rightarrow f(n) = 0 \quad \forall n \in \mathbb{Z} \Rightarrow f \equiv 0$

SHANNON

$$f(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2 \qquad f \in \mathcal{M}(\mathbb{R})$$

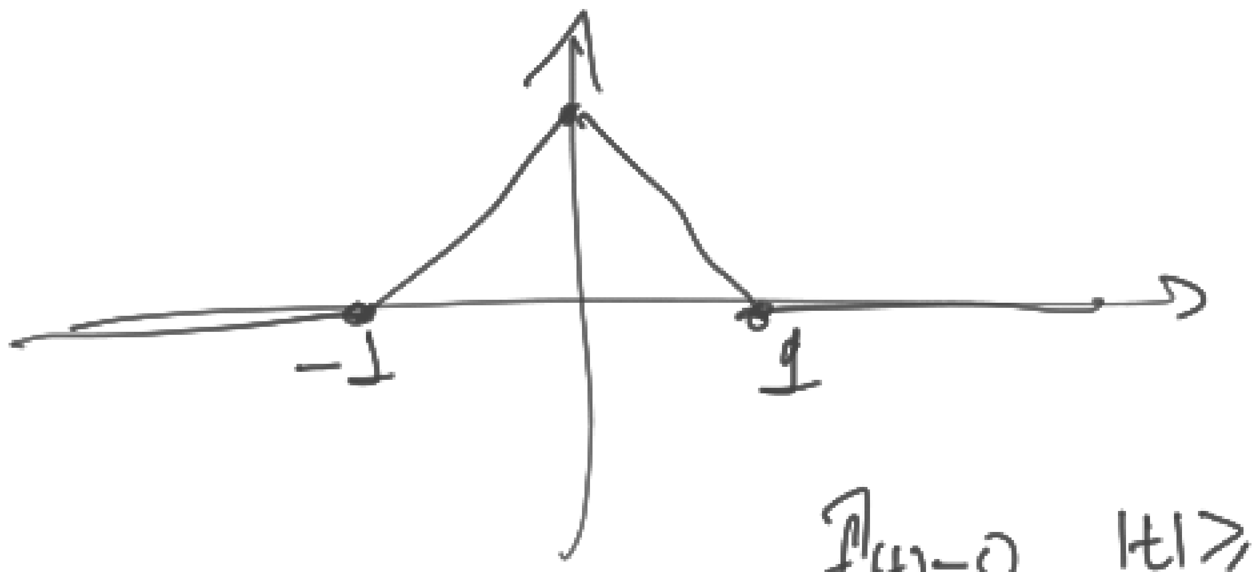
- $|z| \leq 1 \Rightarrow |f(z)| \leq M$.
- $|z| \geq 1 \Rightarrow |f(z)| \leq \frac{|\sin \pi z|^2}{\pi^2} = \left|\frac{e^{i\pi z} - e^{-i\pi z}}{2i\pi}\right|^2$

$|e^z| \leq e^{|z|}$

$$\leq \frac{\left(|e^{i\pi z}| + |e^{-i\pi z}|\right)^2}{4\pi^2} \leq \frac{\left(2e^{\pi|z|}\right)^2}{4\pi^2} = \frac{e^{2\pi|z|}}{\pi^2}$$

$\Rightarrow f$ has exponential type $2\pi \xRightarrow{\text{P.W.}} \operatorname{supp} \hat{f} \subset [-1, 1]$

$f(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \to \hat{f}(t) = \max\{1-|t|, 0\}$



$\hat{f}(t) = 0, \quad |t| \geq 1$

$f(z) = \frac{\sin \pi z}{\pi z (z-1)} \to \operatorname{supp} \hat{f} \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$